4.3 Index of a subgroup, the Counting Formula

Let H be a subgroup of a group G.

Notation 4.25. The **set** of left cosets of H in G is denoted G/H. (The set of right cosets of H in G is denoted $H\backslash G$.)

Remark 4.26. *Warning:* In general, G/H is a just a set, not a group. We will see that if $G/H = H\backslash G$, then the group operation on G induces a group operation on the set G/H. In this case, H is a *normal subgroup*, and G/H, with the induced operation, is a *quotient group*.

Proposition 4.27. The subgroup $H \leq G$ has the same number of left and right cosets.

Proof. HW05 □

Definition 4.28. The **index** of H in G, denoted $[G : H]$, is the number ($\in \mathbb{N} \cup \{\infty\}$) of left cosets of H in G.

Theorem 4.29. (The Counting Formula) Let $H \leq G$. Then $|G| = |H| \cdot [G : H]$.

Proof. First consider the case where $|G| < \infty$. Since G/H forms a partition of G, and every coset aH contains $|H|$ elements, there are $|G|/|H|$ left cosets in total.

Now suppose $|G| = \infty$. We will check that either $|H|$ or $[G : H]$ must be infinite too. (Note first that $|H|, [G : H]$ are both natural numbers, i.e. ≥ 1.) By way of contradiction, suppose that both $|H|$ and $[G : H] = k$ were finite. From each of the $k = [G : H]$ left cosets of H, we can pick a representative; this gives us a set of representatives $\{a_1, a_2, \ldots, a_k\}$, with each from a different coset. Then $G = \bigcup_{i=1}^{k} a_i H$ contains $k \cdot |H| < \infty$ elements, which is a contradiction. □

Corollary 4.30.

- For $H \leq G$, $|H|$ divides $|G|$, i.e. $|H| \mid |G|$.
- For $g \in G$, $|g|$ divides $|G|$, i.e. $|g| \mid |G|$.

This is useful when classifying groups of a particular finite order.

Example 4.31. Let $|G| = p$ where p is a prime number. For any non-identity $a \in G$. $G = \langle a \rangle$. Therefore there is only one isomorphism (equivalence) class of groups of order p prime.

Corollary 4.32. Let $\varphi : G \to G'$ be a homomorphism.

- $[G : \ker \varphi] = |\operatorname{img} \varphi|$ (Therefore $|G| = |\ker \varphi||\operatorname{img} \varphi|$.)
- $|\ker \varphi| \mid |G|$
- $|\operatorname{img} \varphi| \mid |G|$ and $|\operatorname{img} \varphi| \mid |G'|$.

Exercise 4.33. HW04 Let $\varphi : G \to G'$ be a group homomorphism. Suppose that $|G| = 18$ and $|G'| = 15$, and that φ is not the trivial homomorphism. What is the $|\ker \varphi|$?

Example 4.34. Recall that $A_n = \ker \operatorname{sgn}$, where $\operatorname{sgn} : S_n \to \{\pm 1\}$ is the sign homomorphism. Therefore the order of $A_n = \dfrac{|S_n|}{2} = \dfrac{n!}{2}$.

Proposition 4.35. If $K \leq H \leq G$, then $[G : K] = [G : H][H : K]$.

Proof. (Proof sketch.) First consider the case where both indices on the right side are finite, and consider partitions of G and H by cosets of H and K, respectively. Then consider the case where at least one of the indices on the right is infinite, and show that $[G : K]$ has to be infinite as well. □

Proposition 4.36. If $\varphi : G \to G'$ is an isomorphism, then the inverse *set* map is also an isomorphism.

Proof. HW05 □