4.8 Correspondence Theorem

Let $\varphi : G \to \mathcal{G}$ be a group homomorphism, and let $H \leq G$.

We may **restrict** φ to a homomorphism

$$\begin{aligned} \varphi|_H : H &\to \mathcal{G} \\ h &\mapsto \varphi(h) \end{aligned}$$

- $\ker(\varphi|_H) = (\ker \varphi) \cap H$
- $\operatorname{img}(\varphi|_H) = \varphi(H)$

Remark 4.84. Since $\varphi|_H$ is a homomorphism, the order of the image $\varphi(H)$ divides both $|H|$ and $|\mathcal{G}|$. If $|H|$ and $|\mathcal{G}|$ have no common factors, then $H \leq \ker \varphi$.

Example 4.85. Recall A_n is the kernel of the sign homomorphism $\sigma : S_n \to \pm 1$.

Let q be a permutation with odd order, and let $H = \langle q \rangle$. Then $H \leq A_n$.

Proposition 4.86. Let $\varphi : G \to \mathcal{G}$ be a homomorphism with kernel K. Let $\mathcal{H} \leq \mathcal{G}$, and let $H = \varphi^{-1}(\mathcal{H})$.

1. Then $K \leq H \leq G$. *(A chain of subgroups.)*
2. If $\mathcal{H} \trianglelefteq \mathcal{G}$, then $H \trianglelefteq G$.
3. If φ is surjective and $H \trianglelefteq G$, then $\mathcal{H} \trianglelefteq \mathcal{G}$.

Proof.
1. *Check carefully; note that φ^{-1} means* preimage*.*
2. Suppose $\mathcal{H} \trianglelefteq \mathcal{G}$. Let $x \in H, g \in G$. Then $\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1} \in \mathcal{H}$ because $\mathcal{H} \trianglelefteq \mathcal{G}$.
3. Suppose φ is surjective and $H \trianglelefteq G$. Let $a \in \mathcal{H}, b \in \mathcal{G}$. Since φ is surjective, there exist elements $x \in H, y \in G$ such that $\varphi(x) = a$, $\varphi(y) = b$. Since H is normal, $yxy^{-1} \in H$, so $\varphi(yxy^{-1}) = bab^{-1} \in \mathcal{H}$.

□

Example 4.87. Consider $\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$. Since $\mathbb{R}^\times$ is abelian, $\mathbb{R}^\times_{>0} \trianglelefteq \mathbb{R}^\times$. The preimage under $\det$ of the positive reals is the set of invertible matrices with positive determinant, and is therefore a normal subgroup of $GL_n(\mathbb{R})$.

Theorem 4.88. (The Correspondence Theorem) Let $\varphi : G \to \mathcal{G}$ be a *surjective* group homomorphism with kernel K. Then there is a bijective correspondence

$$\{\text{subgroups of } G \text{ that contain } K\} \leftrightarrow \{\text{subgroups of } \mathcal{G}\}.$$

The correspondence is given by

$$\mathcal{H} \rightsquigarrow \varphi^{-1}(\mathcal{H}).$$

Suppose H and $\mathcal{H}$ are corresponding subgroups. Then:

- $H \trianglelefteq G$ if and only if $\mathcal{H} \trianglelefteq \mathcal{G}$.
- $|H| = |\mathcal{H}||K|$.

Proof. Here are the things to check:

1. $\varphi(H)$ is a subgroup of $\mathcal{G}$
2. $\varphi^{-1}(\mathcal{H})$ is a subgroup of G, and it contains K
3. $\mathcal{H} \trianglelefteq \mathcal{G}$ if and only if $\varphi^{-1}(\mathcal{H}) \trianglelefteq G$
4. *Bijectivity of the correspondence:* $\varphi(\varphi^{-1}(\mathcal{H})) = \mathcal{H}$ and $\varphi^{-1}\varphi(H) = H$.
5. $|\varphi^{-1}(\mathcal{H})| = |\mathcal{H}||K|$.

□

Exercise 4.89. Let $\varphi : G \to G'$ be a surjective homomorphism between finite groups. Suppose $H \leq G$ and $H' \leq G'$ correspond to each other under the bijection in the Correspondence Theorem. Prove that $[G : H] = [G' : H']$.

Exercise 4.90. Let C_{12} be generated by x and let C_6 be generated by y. Consider the surjective homomorphism $\varphi : C_{12} \to C_6$ determined by $x \mapsto y$. Explicitly write down the correspondence between subsets given by the Correspondence Theorem. *If you are claiming a group G has k subgroups, you must explain (briefly) why you've found all of them.*

Example 4.91. Here's a diagram of the subgroup structure of S_3: