5 Symmetries of plane figures

5.1 Distance in $\mathbb{R}^2$

We can think of the additive group $\mathbb{R}^2$ as a group of vectors or a group of points in the plane. In any case, Euclidean distance gives us a notion of distance between two elements $\vec{x}, \vec{y} \in \mathbb{R}^2$:

$$d(\vec{x}, \vec{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

This distance function is actually induced by the dot product, as follows.
Recall that for $\vec{v}, \vec{w} \in \mathbb{R}^2$, the *dot product* of $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2.$$

The length of the vector $\vec{v}$, or the *norm* of $\vec{v}$ is given by

$$\|\vec{v}\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2}.$$

Given vectors $v, w \in \mathbb{R}^2$ (thought of as points in $\mathbb{R}^2$), the distance between v and w is

$$d(v, w) = \|w - v\| = \|v - w\|.$$

Now consider a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$. If we choose choose a basis for the domain and codomain, we can write A as a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Let $\vec{a}_1$ denote the first column vector and let $\vec{a}_2$ denote the second column vector.

Exercise 5.1. Check that $Ae_i = a_i$ for $i = 1, 2$.

Any vector $\vec{v} \in \mathbb{R}^2$ can be written as a linear combination of the standard basis vectors e_1 and e_2 (because $\{e_1, e_2\}$ is a *basis*):

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 e_1 + v_2 e_2.$$

Since A is a *linear map*, we have

$$A\vec{v} = A(v_1 e_1 + v_2 e_2) = v_1 A e_1 + v_2 A e_2 = v_1 a_1 + v_2 a_2.$$

In other words, the linear map A is determined by its value on the basis vectors e_1 and e_2.

5.2 The Orthogonal Group $O(2)$

When does a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ preserve distances, i.e.

$$d(x, y) = d(Ax, Ay)?$$

Intuitively, this should be the linear maps that rigidly rotate or reflect the plane, without any squeezing or stretching. In particular, this means that the standard basis vectors e_1 and e_2 are sent to vectors a_1 and a_2 which are still unit vectors that are orthogonal to each other.

Definition 5.2. Two vectors $a_1, a_2 \in \mathbb{R}^2$ are orthonormal if

- $a_1 \cdot a_2 = 0$ (i.e. $a_1 \perp a_2$)
- $\|a_1\| = \|a_2\| = 1$ (i.e. a_1 and a_2 are *unit vectors*, i.e. vectors of length 1)

Definition 5.3. A matrix $A = [a_1\ a_2]$ is **orthogonal** if its columns $\{a_1, a_2\}$ are orthonormal.

Definition 5.4. The **orthogonal group** $O(2)$ is the group of orthogonal 2×2 matrices.

Exercise 5.5. Prove that if A is orthogonal, then A preserves distances.

It turns out that the converse is also true: 2×2 matrices that preserve distance are orthogonal.
We now discuss what $O(2)$ looks like as a group. Let

$$\rho_\theta := \begin{bmatrix} \cos\theta & -\sin(\theta) \\ \sin\theta & \cos\theta \end{bmatrix}$$

denote rotation by θ about the origin (counter-clockwise, of course). Let

$$\tau = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

denote reflection across the e_1-axis.

Fact 5.6. Any matrix in $O(2)$ is either of the form ρ_θ or $\rho_\theta\tau$.

- The set of orthogonal matrices that are just simple rotations $\{\rho_\theta \mid \theta \in [0, 2\pi)$ is the set of *orientation-preserving* orthogonal matrices. In other words, the matrix takes the "front" of the plane to the "front".
- On the other hand, the set of orthogonal matrices that are rotations composed with a reflection are *orientation-reversing*; they take the "front" of $\mathbb{R}^2$ to the "back".

This fact tells us that orthogonal actions such as reflection about a line that is *not* the e_1-axis can be written as the product of a rotation and the reflection τ.
Here are two important subgroups of $O(2)$:

- $S^1 \cong$ the set of rotations = $\{\rho_\theta \mid \theta \in [0, 2\pi)$ (We originally defined S^1 as a subgroup of $\mathbb{C}^\times$; notice that there is an isomorphism between this group of rotation matrices and S^1 the subgroup of $\mathbb{C}^\times$.)
- $\mathbb{Z}/2\mathbb{Z} \cong \langle\tau\rangle$, the order 2 cyclic subgroup generated by the reflection τ. (Notice that $\tau = \tau^{-1}$.)

Exercise 5.7. Prove that $S^1 \trianglelefteq O(2)$. Solution: S^1 has index 2.

5.3 $O(2)$ is a semi-direct product

Temporarily write $N = S^1$ and $H = \mathbb{Z}/2\mathbb{Z}$. Even though Fact 5.6 tells us that $G = NH$ as a set, $O(2)$ is **not** the direct product of the subgroups N and H. This is because the elements of N and H don't commute! We already saw this when we looked at dihedral groups, which are themselves subgroups of $O(2)$: for any rotation ρ,

$$\rho\tau\rho\tau = 1 \implies \tau\rho\tau = \rho^{-1}.$$

Therefore if $\rho \neq \rho^{-1}$, then conjugation by τ does not fix ρ.
However, all is not lost, because $N \trianglelefteq O(2)$. It turns out that $O(2)$ is a *semi-direct product* of S^1 and $\mathbb{Z}/2\mathbb{Z}$.

Definition 5.8. Let G be a group, and let $N, H \leq G$. If $N \trianglelefteq G$, $G = NH$, and $N \cap H = \{1\}$, then G is a **semi-direct product** of N and H. This is written

$$G = N \rtimes H.$$

Remark 5.9. This is not a definition I necessarily want you to memorize; I just want to show you how similar the conditions are to those in the proposition characterizing product groups.

The underlying set of $N \rtimes H$ is still the Cartesian product $N \times H$; however, multiplication is *twisted* by conjugation. Let $(n, h), (m, k) \in N \times H$ (as a set). Then their product in the semi-direct product $N \rtimes H$ is

$$(n, h) \cdot (m, k) = (nc_h(m), hk)$$

where $c_h(m) = hmh^{-1} \in N$ is the conjugation of m by h. (This is where we need N to be normal in G.

The multiplication formula might seem unnatural, but the following computation should hopefully convince you that, if you already know N, H were subgroups of a bigger group G where we already have multiplication, then the formula above is very natural.

Recall that $G = NH$, so every element can be written in the for nh for $n \in N$, $h \in H$. Let $n_1h_1, n_2h_2 \in NH = G$. Their product in G is

$$(n_1h_1)(n_2h_2) = n_1h_1n_2h_2.$$

We wish to move the n_2 to the left of the h_1 in order to write the product in the form nh. To do this, we can rewrite our product:

$$n_1h_1n_2h_2 = n_1h_1n_2(h_1^{-1}h_1)h_2 = n_1(h_1n_2h_1^{-1})h_1h_2 = n_1c_{h_1}(n_2)h_1h_2 \in NH.$$

In other words, **the cost of commuting n_2 past h_1 is conjugation by h_1.**

Fact 5.10. $O(2) = S_1 \rtimes \mathbb{Z}/2\mathbb{Z}$.

Let $\rho_\alpha a$ and $\rho_\beta b$ be two elements in $O(2)$, where $\rho_\alpha, \rho_\beta \in S_1$ and $a, b \in \{1, \tau\} = \mathbb{Z}/2\mathbb{Z}$. Then multiplication in $O(2)$ is given by

$$(\rho_\alpha a)(\rho_\beta b) = \rho_\alpha c_a(\rho_\beta)ab.$$

Notice that if $a = 1$, then conjugation by a does nothing (and we might as well have written $\rho_\alpha a \rho_\beta b$ as $\rho_\alpha \rho_\beta b$, which is already in the form we like).

On the other hand, if $a = \tau$, then $c_a(\rho_\beta) = \rho_\beta^{-1} = \rho_{-\beta}$.

Example 5.11. To drive this idea home, let's compute the product of these two orientation-reversing elements of $O(2)$:

$$\begin{aligned}(\rho_\alpha\tau)(\rho_\beta\tau) &= \rho_\alpha(\tau\rho_\beta\tau^{-1})(\tau\tau) \\ &= \rho_\alpha\rho_{-\beta}\tau^2 \\ &= \rho_{\alpha-\beta}.\end{aligned}$$

The result is a rotation by an angle $\alpha - \beta$. (Try it!)