5.4 Isometries of the plane

Definition 5.12. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an **isometry** if it preserves distances:

$$d(p, q) = d(f(p), f(q)) \quad \text{for all points } p, q \in \mathbb{R}^2$$

Let $\mathrm{Isom}(\mathbb{R}^2)$ denote the group of isometries of $\mathbb{R}^2$.

We think of isometries of $\mathbb{R}^2$ as **symmetries** of the plane. In particular, we can study the symmetries of the plane by studying symmetries of **plane figures**. These are subsets of the plane, such as the drawing of a stick figure. (See the book for pictures of various symmetries of plane figures.)

Fact 5.13. $\mathrm{Isom}(\mathbb{R}^2)$ is *generated* by the following elements. Let x be a point in $\mathbb{R}^2$:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- **Translations**: for a translation vector $v \in \mathbb{R}^2$, and a point $x \in \mathbb{R}^2$,

$$t_v(x) = x + v.$$

- **Rotations**: for an angle $\theta \in S^1$ and a point $x \in \mathbb{R}^2$,

$$\rho_\theta(x) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- **Reflection across the e_1-axis**: for a point $x \in \mathbb{R}^2$,

$$\tau(x) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Remark 5.14. Warning: The points in $\mathbb{R}^2$ are those being moved around by the isometries. The translations vectors $v \in \mathbb{R}^2$ are **not** the same as the points in the plane. You should think of them as velocity vectors.

Proposition 5.15. The subgroup of translations $T = \{t_v \mid v \in \mathbb{R}^2\} \leq \mathrm{Isom}(\mathbb{R}^2)$ is normal.

Proof. For any $g \in \mathrm{Isom}(\mathbb{R}^2)$, we need to show that $g t_v g^{-1}$ is also a translation. It suffices to just check the cases where g is a generator, since every isometry is a composition of these.

First check that T is a subgroup; then the conjugation of t_v by any translations is necessarily also a translation.

Next, let $g = \rho_\theta$, and let $c = \cos\theta$ and $s = \sin\theta$. The rotation matrix for ρ_θ and $\rho_\theta^{-1} = \rho_{-\theta}$ are

$$\rho_\theta = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \qquad \text{and} \qquad \rho_{-\theta} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

respectively. (Use the fact that cosine is an even function, and sine is an odd function.) Compute that

$$\rho_\theta t_v \rho_{-\theta} = t_{\rho_\theta v}.$$

Third, let $g = \tau$. Compute that

$$\tau t_v \tau = t_{\tau v}.$$

□

Exercise 5.16. HW07 We used $\mathbb{R}^2$ to describe the points on the plane. We could equivalently use $\mathbb{C}$, the complex plane. Since we use the same notion of distance for points in the complex plane, as metric spaces, $\mathbb{R}^2$ is the same as $\mathbb{C}$. Write formulas for the generators of $\mathrm{Isom}(\mathbb{C})$ in terms of the complex variable $z = x + iy$.

Exercise 5.17. HW07 Prove that a conjugate of a glide reflection in $\mathrm{Isom}(\mathbb{R}^2)$ is also a glide reflection, and that the glide vectors have the same length.