5.6 Discrete subgroups of $\mathrm{Isom}(\mathbb{R}^2)$

Let $H \leq \mathrm{Isom}(\mathbb{R}^2)$.

- H contains an arbitrarily small translation if, for any $\varepsilon > 0$, there is a translation $t_v \in H$ such that $0 < |v| < \varepsilon$.
- Similarly, H contains arbitrarily small rotations if, for any $\varepsilon > 0$, there is a rotation $\rho_\theta \in H$ such that $0 < |\theta| < \varepsilon$.

Definition 5.21. A group G of isometries of the plane (i.e. $G \leq \mathrm{Isom}(\mathbb{R}^2)$) is **discrete** if it does not contain arbitrarily small translations or rotations.

In other words, G is **discrete** if there exists a real number ε such that

- if $t_v \in G$ and $v \neq 0$ (i.e. $t_v \neq \mathrm{id}$), then $|v| > \varepsilon$, and
- if $\rho_\theta \in G$, where $\theta \in [-\pi, \pi)$, then $|\theta| \geq \varepsilon$.

Given a discrete group of isometries $G \leq \mathrm{Isom}(\mathbb{R}^2)$, we will study the following subgroups:

- the **translation group** $L \leq G$, a subgroup of the group of translations $T \leq \mathrm{Isom}(\mathbb{R}^2)$
- the **point group** $\overline{G}$, a subgroup of the orthogonal group $O(2) \leq \mathrm{Isom}(\mathbb{R}^2)$.

Exercise 5.22. Explain why, in the setup above, $G \cong L \rtimes \overline{G}$.

The following theorem classifies all possible translation groups:

Theorem 5.23. Every discrete subgroup $L \leq T \cong \mathbb{R}^2$ is one of the following:

- the zero group: $L = \{0\}$
- the set of integer multiples of a nonzero vector a: $L = \mathbb{Z}a$
- the set of integer combinations of two linearly independent vectors a and b: $L = \mathbb{Z}a + \mathbb{Z}b$. *Groups of this type are called* ***lattices****.*

Proof. We will use the following Lemma, which describes some fairly intuitive geometric properties of discrete sets of points/vectors in the plane.

Lemma 5.24. Let D be a discrete set of points in the plane, i.e. there is some $\varepsilon > 0$ such that, for all points $p \neq q$ in D, $d(p, q) \geq \varepsilon$.

(A) A bounded region of the plane contains only finitely many points in D.

(B) If $D \neq \{0\}$, then it contains a non-origin point of minimal distance from the origin.

Recall the difference between infimum and minimum from Mat 108.

Remark 5.25. When we say *minimal length* vector in $L \leq \mathbb{R}^2$, we mean a *nonzero* vector of minimal length.

We now work in cases, at times describing the elements of L as points or as vectors, as needed in context.

Case 0: L is the trivial subgroup Let L be a discrete subgroup L of $\mathbb{R}^2$. If $L = \{0\}$, then we are done.

Case 1: L lies on a line through the origin Now suppose L is not just the trivial subgroup, and all points lie on a line ℓ. (This line must necessarily go through the origin, which is the identity element in L.) Let a be a minimal length vector in L; we want to show that $L = \mathbb{Z}a$. Suppose by way of contradiction that there is some vector b that is not an integer multiple of a. Let ka be a multiple of a that is closest to b. Then $b - ka$ is a nonzero vector of length shorter than a. This is a contradiction to the minimality of a.

Case 2: L is none of the above We now use the same idea we used in Case 1, but obtain two "short" vectors that are linearly independent. First let a be a minimal length vector. Since L does not lie on a line, $L - \mathbb{Z}a$ is nonempty and still discrete, so we can find a vector b that is minimal length in $L - \mathbb{Z}$. We want to show that $L = \mathbb{Z}a + \mathbb{Z}b$. Suppose there is some vector $c \in L$ that is not a linear combination of a and b. Then c lies inside a parallelogram whose vertices are the lattice $\mathbb{Z}a + \mathbb{Z}b$. Let $ia + jb$ be a lattice point closest to c. Then the vector $c - (ia + jb)$ is shorter than b, which contradicts the minimality of b in $L - \mathbb{Z}a$. *(Draw a picture!)*

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