The following proposition basically tells us that if we view $p \in L$ as the result of translating the origin by t_p, that if $\bar{g}$ is in the point group, the point $\bar{g}(p)$ will also be a point in the lattice L (i.e. a translation of 0 by something in L).

Proposition 5.26. Let G be a discrete subgroup of $\mathrm{Isom}(\mathbb{R}^2)$. Let a be an element of its translation group L, and let $\bar{g}$ be an element of its point group $\overline{G}$. Then $\bar{g}(a) \in L$.

Proof. To show that $\bar{g}(a) \in L$, we just need to show that $t_{\bar{g}(a)} \in G$. Indeed, we showed previously that $t_{\bar{g}(a)} = g t_a g^{-1} \in G$. □

It's worth taking some time to really absorb what the above proposition is saying, while looking at a wallpaper pattern. They key to fully understanding the proposition is to make sure you're clear on the separation between *isometries* of $\mathbb{R}^2$ (which are symmetries of the wallpaper) and the points in the plane themselves (which we get from picking a particular point on the wallpaper and moving it around).

With the above proposition, we can now describe point groups of discrete subgroups $G \leq \mathrm{Isom}(\mathbb{R}^2)$ by studying symmetries of lattices Λ that take the form of a rotation or reflection in $O(2)$.

The following theorem classifies all possible point groups:

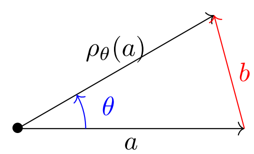
Theorem 5.27 (Crystallographic Restriction)**.** Let Λ be a **discrete** subgroup of $\mathbb{R}^2$, and let $\mathrm{Sym}(\Lambda) \leq \mathrm{Isom}(\mathbb{R}^2)$ denote the group of symmetries of Λ.

Let $H \leq O(2) \cap \mathrm{Sym}(\lambda)$, and suppose that $\Lambda \neq \{0\}$. Then

1. every rotation in H has order 1,2,3,4, or 6, and

2. H is one of the groups C_n or D_n, where $n \in \{1, 2, 3, 4, 6\}$.

Proof. It suffices to prove (a). Let ρ_θ be a rotation in H. Let $a \in \Lambda$ be a *minimal length* translation vector $t_a \in \mathrm{Sym}(\Lambda)$. Then $\rho_\theta t_a = t_{\rho_\theta(a)} \in \mathrm{Sym}(\Lambda)$, so $\rho_\theta(a) \in \Lambda$. Let $b = \rho(a) - a$:

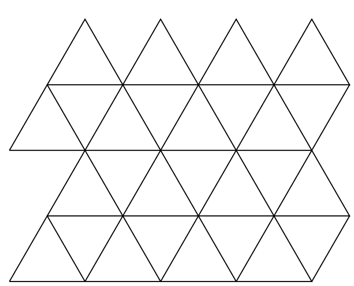


From the figure, we see that $\|b\| < \|a\|$ if $\theta < \pi/3$. So by minimality of a, we must have $\theta \geq \pi/3$. Therefore $|\rho_\theta| \leq 6$.

We can easily construct lattices Λ with symmetries ρ_θ of order $1, 2, 3, 4, 6$. *Try this yourself.*

It remains to show that $\theta = 2\pi/5$ is *impossible*. Let $\phi = 2\pi/5$. If $\rho_\phi \in H$, then $b = \rho_\phi^2(a) + a \in \Lambda$ as well. But then b is shorter than a, which again contradicts the minimality of a. *Use trigonometric functions to prove this for yourself!* $e^{\frac{4\pi i}{5}} \approx -.81 + .59i$ □

Exercise 5.28. HW08 Let G denote the group of symmetries of the following **infinite** wallpaper pattern P constructed from equilateral triangles of side length 1:



(a) Determine the point group $\overline{G}$ of G, and find the index in G of the subgroup of translations L.

(b) Find translation vectors $a, b \in \mathbb{R}^2$ realizing L as the lattice $\mathbb{Z}a + \mathbb{Z}b$.