6.2 Orbit-stabilizer theorem

Proposition 6.17. Suppose a group G acts on a set S. Let $s \in S$. Let G_s denote the stabilizer of s, and let O_s denote the orbit of s.

There is a bijective map (of sets!)

$$\varepsilon : G/G_s \to O_s$$
$$[aG_s] \mapsto as$$

that respects the action of G on both sides, i.e.

$$\varepsilon(g[C]) = g\varepsilon([C])$$

for every coset C and every element $g \in G$. (We say that the map ε is G-**equivariant**.)

Proof. For the purposes of this proof, we let $H = G_s$.

First, we need to show that ε is well-defined. Suppose $aH = bH$; we need to show that $as = bs$. Since $a \in bH$, there is some $h \in H$ such that $a = bh$. Since $h \in H = G_s$ fixes s, $as = bhs = bs$.

Second, we show that ε is injective. If $\varepsilon(aH) = \varepsilon(bH)$, then $as = bs$, so $b^{-1}as = b^{-1}bs = 1s = s$. Then $b^{-1}a \in H$, so $aH = bH$ indeed.

Third, we show that ε is surjective. If $s' \in O_s$, then there is some $g \in G$ such that $s' = gs$. Then $\varepsilon(gH) = gs = s'$.

Finally, we need to check that ε is G-equivariant. Let $g \in G$, and let $[aH] \in G/H$. Then

$$\varepsilon(g[aH]) = \varepsilon([gaH]) = gas = g(as) = g\varepsilon([aH]).$$

□

Example 6.18. Here are some examples illustrating the Orbit-Stabilizer Theorem for transitive actions.

1. Consider the action of D_5 on the vertices V of a regular pentagon. Let $v \in V$ and let H be the stabilizer of v. Then thre is a bijection
$$\varepsilon : D_5/H \to V$$
since the orbit of v is all of V.

2. Consider $\mathrm{Isom}(\mathbb{R}^2) \curvearrowright \mathbb{R}^2$. The stabilizer of the origin is O_2. The orbit of the origin is the entire plane. So there is a bijection between $T \cong \mathrm{Isom}(\mathbb{R}^2)/O(2)$ and $\mathbb{R}^2$. (Recall that T was the normal subgroup of translations.)

3. Let $\mathcal{L}$ denote the set of all lines in $\mathbb{R}^2$. There is an induced action by $\mathrm{Isom}(\mathbb{R}^2)$. For $L \in \mathcal{L}$, let H_L denote the stabilizer of L. Then $\mathrm{Isom}(\mathbb{R}^2)/H_L \leftrightarrow \mathcal{L}$.

Exercise 6.19. On the other hand, consider the non-transitive action of $H = \langle \tau \rangle \leq D_5$ on the vertices V of a pentagon. There are three orbits. Exhibit the bijective map ε for all three of these orbits.

Exercise 6.20. Exhibit the bijective map ε from the orbit-stabilizer theorem explicitly, for the case where G is the dihedral group D_4 and S is the set of vertices of a square.

The Orbit-Stabilizer Theorem is very often used to count things. Recall that the Counting Formula tells us

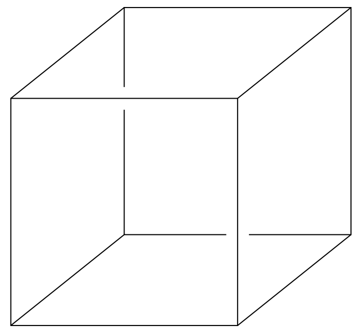
$$|G| = |H||G/H|.$$

In terms of group actions, we have yet another version of the counting formula.

Observation 6.21 (Counting Formula)**.** Let S be a finite set on which G acts. Let $s \in S$. By the Orbit-Stabilizer Theorem,

$$|G| = |G_s||O_s|.$$

Here is an example that illustrates we can use this formula to determine the size of a symmetry group. Consider a cube:



Question 6.22. How big is the set of orientation-preserving symmetries of the cube?

First, a couple of remarks:

1. To rephrase this in terms of abstract algebra, we first note that the set of symmetries is actually a group. So, we can rephrase this question as follows. Let G be the group of orientation-preserving symmetries of the cube. What is $|G|$, the order of the group G?
2. This the 3D analogue to the symmetries of plane figures, such as a square. The symmetries must be isometries of $\mathbb{R}^3$.
3. **Orientation-preserving** means that you can't reflect the cube through a plane; we really want to only consider symmetries that you can physically perform on a real-life cube, such as a die.
4. If we're looking at a solid object in real life (i.e. not an infinite 3D object), then the group of orientation-preserving symmetries consists only of rotations. So, the book will call these **rotational symmetries**

In order to answer this, one could try to count all the symmetries. Or, one could focus on, say, the set of faces. That is, there is clearly a natural action of G on the set of 6 faces of a cube. Let f be a particular face. The only actions I can perform that preserve a given face are the four rotations about the line normal to that face. Therefore $|G_f| = 4$. The orbit of f is all six faces of the cube, so $|O_f| = 6$. Then by the Orbit-Stabilizer Theorem and Counting Formula, we know $|G| = 24$.

Exercise 6.23. Let G be the set of rotational symmetries of a regular dodecahedron. This is a solid with 12 faces that are all regular pentagons. What is $|G|$?

We can also use algebra to figure out the size of a set that a group acts on, by using the following observation.

Observation 6.24 (Decomposition of S into orbits)**.** Let S be a finite set on which G acts, and let $O_1, O_2, \ldots, O_k$ be the set of orbits. Then

$$|S| = |O_1| + |O_2| + \cdots + |O_k|.$$

More interestingly, by the Counting Formula, for each $i = 1, 2, \ldots, k$, we know that $|O_i|$ **must divide** $|G|$.

This observation is also very useful in many contexts. We'll see this again when we talk about conjugacy classes in groups later on.

Example 6.25. Let G be the set of **rotational symmetries** of a tetrahedron T. (We are only looking at orientation-preserving rigid motions.) Let V, E, F be the set of vertices, edges, and faces, respectively. Observe that $|V| = 6$, $|E| = 4$, and $|F| = 6$.

Pick a vertex v and consider the stabilizer G_v. We can **restrict** the action $G \curvearrowright T$ to an action $G \curvearrowright V$, because we observe that any symmetry of T will necessarily take a vertex to another vertex.

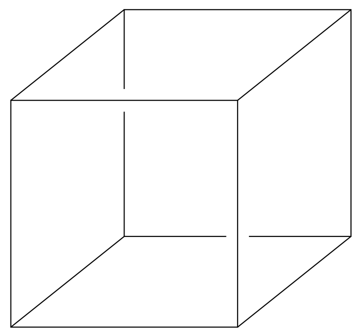
Using geometric reasoning, we see that $G_v \cong \mathbb{Z}/3\mathbb{Z}$ is generated by rotation about the axis going through the vertex v that is normal to the face opposite to v. The action $G_v \curvearrowright V$ has two orbits: v is fixed by G_v, so it's in an orbit on its own; the other three vertices are taken to each other under the action, so they form an orbit. We summarize this by the equation

$$|V| = 4 = 1 + 3.$$

Similarly, any symmetry of T must take an edge to an edge, so we get an induced action $G_v \curvearrowright E$. View T with v at the top of the pyramid with a flat base. The three sloped edges form an orbit, and the three flat edges form another orbit. The orbit decomposition of the set of edges can be summarized as

$$|E| = 6 = 3 + 3.$$

Exercise 6.26. A *cube* is a 3D solid with 6 square faces of equal size:



One example of the cube is the set of points $Q = [0,1]^3 \subset \mathbb{R}^3$.

Let G be the group of **rotational symmetries** of the cube. This is a subgroup of $O(3)$ consisting of *orientation-preserving* symmetries of the cube. [9]

Let V, E, and F denote the sets of vertices, edges, and faces of the cube, respectively. Check for yourself that the size of these sets are

$$|V| = 8 \quad |E| = 12 \quad |F| = 6.$$

(a) Use the counting formula to determine the order of G.

(b) Let G_v, G_e, G_f be the stabilizers of a vertex v, and edge e, and a face f of the cube. Determine the formulas of the form

$$|S| = |O_1| + |O_2| + \cdots + |O_k|$$

(formula 6.9.4 in the text) that represent the decomposition of each of the three sets V, E, F into orbits for each of the subgroups. *Your solution should contain* $9 = 3 \times 3$ *formulas, one for each (group, set) pair. First make sure you are clear on what the group and set in the group action is, in each case!*

We've already talked a bunch about actions *induced* by other actions. Here are two more ways to get induced actions: we can take a subgroup the acting group, or modify the set being acted on.

1. Let $G \curvearrowright S$, and let U be a subset of S. The **stabilizer of the subset** $U \subset S$ is the set H of elements where $gU = U$. Check that H is indeed a subgroup.
   - Observe that then we get an induced action of H on U.
   - We also get an action of G on the orbit of U in the *set of subsets* of S. (See example below.)
2. Let $G \curvearrowright S$, and let $H \leq G$. Then $H \curvearrowright S$.

Example 6.27. Let G be the group of rotational symmetries of the cube. We already computed that there are 24 such symmetries, by considering the action of G on F, the set of 6 faces. From $G \curvearrowright F$, we also get an action of G on *pairs of* faces. There are $\binom{6}{2}$ unordered pairs of faces.

[9]The group of orientation-preserving isometries of $\mathbb{R}^3$ is called $SO(3)$.