Lecture 13

Let's do $\otimes$-Hom carefully.

Note A_R, ${}_RB_S$, C_S

$\Rightarrow$ $\mathrm{Hom}_S({}_RB_S, C_S)$ is right R-module:

$f : {}_RB_S \longrightarrow C_S \in \mathrm{Hom}_{\mathrm{Mod}_S}({}_RB_S, C_S)$

Not left R-mod – actually left R^{op}-mod:

$(r\cdot f)(b) = f(rb) \in C$

Then $(r_1 r_2)\cdot f : b \longmapsto f((r_1 r_2)b)$

while $r_1\cdot(r_2\cdot f) : b \longmapsto (r_2\cdot f)(r_1 b)$

$= f(r_2 r_1 b)$

$\Rightarrow$ define $(f\cdot r)(b) = f(rb)$

$\Rightarrow (f\cdot r_1 r_2)(b) = ((f\cdot r_1)\cdot r_2)(b)$.

Adjoint Isom:

<u>thm.</u> Let R, S be rings. Restate more carefully.

Given modules A_R, ${}_R B_S$, ${}_{\cancel{S}}C_S$ in the appropriate categories.

There is an isomorphism of abelian groups

$$\tau_{A,B,C} : \mathrm{Hom}_S(A \otimes_R B, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)). \qquad \ast$$

$$[f : A \otimes_R B \longrightarrow C] \longmapsto \left(f_a^* : B \longrightarrow C \right)_{a \in A}$$

$$b \longmapsto f(a \otimes b$$

the following agrees w/ the above.

<u>Indeed</u>, if we fix two out of three in $\{A, B, C\}$, the maps $\tau_{A,B,C}$ constitute natural <u>isomorphisms</u>

① $\mathrm{Hom}_S(- \otimes_R B, C) \longrightarrow \mathrm{Hom}_R(-, \mathrm{Hom}_S(B, C))$

$\boxed{② \ \mathrm{Hom}_S(A \otimes_R -, C) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(-, C))}$

③ $\mathrm{Hom}_S(A \otimes_R B, -) \longrightarrow \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, -))$

partial proof

<u>Proof</u> that $\tau_{A,B,C}$ is an $\cong$ in Ab:

① $\tau_{A,B,C}$ is a $\mathbb{Z}$-map ($\mathbb{Z}$-homomorphism)

Let $f, g : A \otimes_R B \longrightarrow C$. Then $f+g : A \otimes_R B \longrightarrow C$

$$a \otimes b \longmapsto f(a\otimes b) + g(a \otimes b)$$

- $\tau_{A,B,C}(f) = f^* = \left(f_a^* : B \to C \right)_{a \in A}$ $\quad f_a^*(b) = f(a \otimes b)$

- $\tau_{A,B,C}(g) = g^* = \left(g_a^* : B \to C \right)_{a \in A}$ $\quad g_a^*(b) = g(a \otimes b)$

- $\tau_{A,B,C}(f+g) = (f+g)^* = \left((f+g)_a^* : B \to C \right)_{a \in A}$

where

$$(f+g)_a^* : B \to C$$
$$b \mapsto (f+g)(a \otimes b) = f(a\otimes b) + g(a \otimes b) \quad \text{indeed.}$$

So indeed $\tau(f+g) = \tau(f) + \tau(g)$

② τ is injective: $\quad$ show $\ker \tau = 0$

<u>Note:</u> $\tau(f) = f^* = (f_a)_{a \in A}$.

If $\tau(f)_a = 0$ for all $a \in A$, then

$$0 = \tau(f)_a(b) = f(a \otimes b) \text{ for all } a \in A,\ b \in B.$$

$\Rightarrow f = 0$ since it vanishes on all generators of $A \otimes B$!

③ τ is surjective:

Let $F \in \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$.

Then
$$\begin{aligned} F : A &\longrightarrow \mathrm{Hom}_S(B, C) \\ a &\longmapsto [F_a : B \to C] \end{aligned}$$

Define
$$\begin{aligned} \varphi : A \times B &\longrightarrow C \\ (a, b) &\longmapsto F_a(b) \end{aligned}$$

Want to use:

$$\begin{tikzcd} A \times B \arrow[rr, "h"] \arrow[dr, "\varphi"'] & & A \otimes_R B \arrow[dl, dashed, "\tilde{\varphi}"] \\ & C & \end{tikzcd}$$

Check:

φ is R-biadditive

- additive in A: $F \in \mathrm{Hom}_R(\dots) \Rightarrow$ additive
- additive in B: $F \in \mathrm{Hom}_{..}(\cdots, \mathrm{Hom}_S(B, \cdots)) \Rightarrow$ additive
- Check: $\varphi(ar, b) \overset{?}{=} \varphi(a, rb)$

$$F_{ar}(b) \overset{?}{=} F_a(rb)$$

F is R-hom: $\Rightarrow$
$$\begin{aligned} F : A &\longrightarrow \mathrm{Hom}_S(B, C) \\ ar &\longmapsto \left[\begin{aligned} F_{ar} : B &\longrightarrow C \\ b &\longmapsto F_{ar}(b) \end{aligned}\right] \end{aligned}$$

$$a \longmapsto [F_a : B \to C]$$

$$a \cdot r \longmapsto \left[\begin{aligned} r \cdot F_a : B &\longrightarrow C \\ b &\longmapsto F_a(rb) \end{aligned}\right]$$

Pf. of naturality? (HW06)

tensor-Hom adjunction

If we define the functors

$$F = - \otimes_R B \quad \text{and} \quad G = \mathrm{Hom}_S(B, -),$$

$$F : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_S \qquad G : \mathrm{Mod}_S \longrightarrow \mathrm{Mod}_R.$$

then we can rewrite the above (*) as

$$\tau : \mathrm{Hom}_S(FA, C) \longrightarrow \mathrm{Hom}_R(A, GC)$$

<u>defn</u> Given categories $\mathcal{C}, \mathcal{D}$, an ordered pair (F, G) of covariant functors $\mathcal{C} \to \mathcal{D}$ is an <u>adjoint pair</u> if,

for each pair of objects $C \in \mathcal{C}$, $D \in \mathcal{D}$, there are bijections

$$\tau_{C,D} : \mathrm{Hom}_{\mathcal{D}}(FC, D) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, GD)$$

that are natural transformations in C and D

i.e. $\quad \tau_D = (\tau_{C,D})_{C \in \mathcal{C}}$

and $\quad \tau_C = (\tau_{C,D})_{D \in \mathcal{D}}$

are natural transformations

Why useful?

prop. $A \otimes_R -$ is right exact. (took us one lecture)

new pf.

Let $B \xrightarrow{i} C \xrightarrow{p} D \to 0$ be exact.

WTS $A \otimes B \xrightarrow{id_A \otimes i} A \otimes C \xrightarrow{id_A \otimes p} A \otimes D \longrightarrow 0$ exact.

Because $\mathrm{Hom}_{\mathbb{Z}}(-, C)$ is contravariant left exact,

ISTS for all $X \in Ab$, the following is exact:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R D, X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R C, X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R B, X)$$

Instead, consider this seqn under natural isoms:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R D, X) \arrow[r] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R C, X) \arrow[r] \arrow[d, "\tau"] & \mathrm{Hom}_{\mathbb{Z}}(A \otimes_R B, X) \arrow[d, "\tau"] \\
0 \arrow[r] & \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(D, X)) \arrow[r] & \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(C, X)) \arrow[r] & \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(B, X))
\end{tikzcd}$$

But $\mathrm{Hom}_R(A, -)$ is left covariant exact.

So ISTS

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(D, X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(C, X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(B, X)$$

is exact.

But $\mathrm{Hom}_{\mathbb{Z}}(-, X)$ is contravariant left exact, so this follows from the hypothesis.

Why "Adjoint"? In Vect (ie linear algebra)

If $T: V \to W$ is a linear transformation, then its adjoint is $T^*: W \to V$ such that

$$\underbrace{\langle Tv, w\rangle_W}_{} = \underbrace{\langle v, T^*w\rangle_V}_{} \quad \forall\, v \in V,\ w \in W$$

inner product

In our case, $\mathrm{Hom}_R(-,-)$ and $\mathrm{Hom}_S(-,-)$ are like the innerproducts.

Much more to say here... but maybe we should get to flat modules + such.

defn. A functor $F: {}_R\mathrm{Mod} \to \mathrm{Ab}$ is representable if it is naturally isomorphic to a functor of the form $\mathrm{Hom}_R(A, -)$ for some R-module A.

Too much fun!