

## Lecture 16

Today: Finish up flat modules

- recall relation w/ projective modules  
relation w/ injective modules
- Lazard's Theorem (pg. 512)

A left module over  $R$  is flat iff  
it is the colim of finitely generated  
free left modules.

→ Decided not to prove this in class

- focus: character groups

Done w/ Chp 6.

After holiday, vector spaces.

Then Galois ...

Recall

- free  $\Rightarrow$  projective  $\Rightarrow$  flat
- If  $B_R$  flat,  $D$  divisible ab grp.  
then  $\text{Hom}_\mathbb{Z}(B, D)$  is an injective (left)  $R$ -module.

defn. If  $B$  is a right  $R$ -mod, its character grp  $B^*$  is the left  $R$ -module  $B^* = \text{Hom}_\mathbb{Z}(B, \mathbb{Q}/\mathbb{Z})$

$$rf: b \mapsto f(br)$$

Look familiar? this is the Pontryagin dual (HW04)

Lemma A sequence of right  $R$ -mods

$$0 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 0 \quad \text{is exact}$$

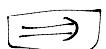
iff the sequence of char. gps is exact.

$$0 \rightarrow C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^* \rightarrow 0$$

Recall For comodules,  $\text{Hom}$  sgn exact  $\Rightarrow$  sgn. exact.

(Though I haven't figured out where in the proof you need commutativity yet)

Pf.



$\mathbb{Q}/\mathbb{Z}$  is injective  $\Rightarrow \text{Hom}_\mathbb{Z}(-, \mathbb{Q}/\mathbb{Z})$  is exact  
(contravariant).



$\text{im } \alpha \subseteq \ker \beta$ : suppose  $\beta \alpha(x) \neq 0$  for some  $x \in A$ .

By HWD4, there is a map  $f: C \rightarrow \mathbb{Q}/\mathbb{Z}$  with  $f\beta \alpha(x) \neq 0$ .

$$= \alpha^* \beta^* f(x)$$

But  $\alpha^* \beta^* = 0$   $\Downarrow$ .

$\ker \beta \subseteq \text{im } \alpha$ :

If  $y \in \ker \beta$  but  $y \notin \text{im } \alpha$ ,

then  $y + \text{im } \alpha \in B/\text{im } \alpha$  is nonzero.

$\Rightarrow \exists$  map  $g: B/\text{im } \alpha \rightarrow \mathbb{Q}/\mathbb{Z}$

with  $g(y + \text{im } \alpha) \neq 0$  (Again by HWD4)

Let  $\pi: B \rightarrow B/\text{im } \alpha$  be the natural map.

Then  $g' = g\pi \in B^*$   $B \xrightarrow{\pi} B/\text{im } \alpha \xrightarrow{g} \mathbb{Q}/\mathbb{Z}$

$$\Rightarrow g'(y) \neq 0.$$

But  $g'(\text{im } \alpha) = \{0\}$  (since it factors thru  $B/\text{im } \alpha$ )

$$\Rightarrow 0 = g'\alpha = \alpha^*(g') \Rightarrow g' \in \text{im } \alpha^* = \text{im } \beta^*.$$

$$\Rightarrow g' = \beta^*(h) \text{ for some } h \in C^*, \text{ ie } g' = h\beta.$$

$$\Rightarrow g'(y) = h\beta(y) = 0 \quad \Downarrow$$

because  $y \in \ker \beta$ !

Observe: We can use the same argument to show exactness at  $A$  and also  $C$ .  $\Rightarrow \alpha$  is injective &  $\beta$  is surjective.



prop (Lambea) A right  $R$ -mod  $B$  is flat iff its char group  $B^*$  is an injective  $R$ -module.

Pf.

$\Rightarrow$  We showed this last time  $\mathbb{Q}/\mathbb{Z} = D$  is divisible.

$\Leftarrow$  Recall to show  $B$  is flat,  $\text{STS } B \otimes_R -$  preserves injectivity.

$B^*$  is injective  $\Rightarrow \text{Hom}_R(-, B^*) = \underbrace{\text{Hom}_R(-, \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}))}$   
is exact. use adjoint isom

Let  $A' \hookrightarrow A$  be injective.

$$\text{Hom}_R(A, B^*) \longrightarrow \text{Hom}_R(A', B^*) \longrightarrow 0.$$

$$\begin{array}{ccc} \parallel & & \parallel \\ \text{Hom}_R(A, \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) & \longrightarrow & \text{Hom}_R(A', \text{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \longrightarrow 0. \end{array}$$

$$\begin{array}{ccc} \downarrow \tau & \curvearrowright & \downarrow \tau \\ \text{Hom}_{\mathbb{Z}}(B \otimes_R A, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \text{Hom}_{\mathbb{Z}}(B \otimes_R A', \mathbb{Q}/\mathbb{Z}) \longrightarrow 0 \end{array}$$

$$\begin{array}{ccc} \parallel & & \parallel \\ (\mathbb{B} \otimes_R A)^* & \longrightarrow & (\mathbb{B} \otimes_R A')^* \longrightarrow 0. \end{array}$$

Then lemma  $\Rightarrow 0 \rightarrow B \otimes_R A' \rightarrow B \otimes_R A$  is exact,  
so  $B$  is flat.