Lecture 16

Today: Finish up flat modules

- recall relation w/ projective modules.
  relation w/ injective modules
- Lazard's Theorem (pg. 512)
  A left module over R is flat iff it is the colim of finitely generated free left modules.

↳ Decided not to prove this in class

- focus: character group

Done w/ Chp 6.
After holiday, vectorspaces.
Then Galois...

Recall

- free $\Rightarrow$ projective $\Rightarrow$ flat

- If B_R flat, D = divisible ab group,
  then $\mathrm{Hom}_{\mathbb{Z}}(B,D)$ is an injective (left) R-module.

defn. If B is a right R-mod, its character group B^* is the left R-module $B^* = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$

$$rf : b \longmapsto f(br)$$

Look familiar? this is the Pontryagin dual (HW04)

Lemma. A sequence of right R-mods

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \qquad \text{is exact}$$

iff the sequence of char. gps is exact.

$$0 \longrightarrow C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^* \longrightarrow 0$$

Recall For comm R, Hom seqn exact $\Rightarrow$ seqn. exact.
(Though I haven't figured out where in the proof you need commutativity yet)

Pf. $\boxed{\Rightarrow}$

$\mathbb{Q}/\mathbb{Z}$ is injective $\Rightarrow$ $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ is exact (contravariant).

$\underline{\operatorname{im}\alpha \subseteq \ker\beta}$: Suppose $\beta\alpha(x) \neq 0$ for some $x \in A$.

By HW04, there is a map $f : C \longrightarrow \mathbb{Q}/\mathbb{Z}$ with $f\beta\alpha(x) \neq 0$.
$= \alpha^*\beta^* f(x)$

But $\alpha^*\beta^* = 0$ ↯.

$\underline{\ker\beta \subseteq \operatorname{im}\alpha}$:

If $y \in \ker\beta$ but $y \notin \operatorname{im}\alpha$,

then $y + \operatorname{im}\alpha \in B/\operatorname{im}\alpha$ is nonzero.

$\Rightarrow \exists$ map $g : B/\operatorname{im}\alpha \longrightarrow \mathbb{Q}/\mathbb{Z}$

with $g(y + \operatorname{im}\alpha) \neq 0$ (Again by HW04)

Let $\pi : B \longrightarrow B/\operatorname{im}\alpha$ be the natural map.

Then $g' = g\pi \in B^*$ $\quad B \xrightarrow{\pi} B/\operatorname{im}\alpha \xrightarrow{g} \mathbb{Q}/\mathbb{Z}$

$\Rightarrow g'(y) \neq 0$.

But $g'(\operatorname{im}\alpha) = \{0\}$ (since it factors thru $B/\operatorname{im}\alpha$)

$\Rightarrow 0 = g'\alpha = \alpha^*(g') \Rightarrow g' \in \ker\alpha^* = \operatorname{im}\beta^*$.

$\Rightarrow g' = \beta^*(h)$ for some $h \in C^*$, ie $g' = h\beta$.

$\Rightarrow g'(y) = h\beta(y) = 0$ ↯.

because $y \in \ker\beta$!

$\underline{\text{Observe}}$: We can use the same argument to show exactness at A and also C. $\rightsquigarrow$ α is injective & β is surjective.

prop. (Lambek) A right R-mod B is flat iff its char. group B^* is an injective R-module.

Pf.

$\boxed{\Rightarrow}$ We showed this last time; $\mathbb{Q}/\mathbb{Z} = D$ is divisible.

$\boxed{\Leftarrow}$ Recall to show B is flat, ISTS $B \otimes_R -$ preserves injectivity.

B^* is injective $\Rightarrow$ $\mathrm{Hom}_R(-, B^*) = \underbrace{\mathrm{Hom}_R(-, \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}))}_{\text{use adjoint isom}}$ is exact.

Let $A' \hookrightarrow A$ be injective.

$$\begin{tikzcd}
\mathrm{Hom}_R(A, B^*) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(A', B^*) \arrow[r] \arrow[d, equal] & 0 \\
\mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \arrow[r] \arrow[d, "\tau"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_R(A', \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})) \arrow[r] \arrow[d, "\tau"] & 0 \\
\mathrm{Hom}_{\mathbb{Z}}(B \otimes_R A, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{Z}}(B \otimes_R A', \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
(B \otimes A)^* \arrow[r] & (B \otimes_R A')^* \arrow[r] & 0
\end{tikzcd}$$

Then lemma $\Rightarrow$ $0 \to B \otimes_R A' \to B \otimes A$ is exact, so B is flat.