# Lecture 18

① finish symm bilinear forms, quadratic forms

② Some alt. forms

③ Tensor + Exterior algebras.

Bonus? Superfast intro to differential forms.

Let $k$ be a commutative ring.

<u>defn.</u> A $k$-algebra $A$ is a <u>graded</u> $k$-algebra if there are $k$-submodules $A^p$ for $p \geq 0$ such that

① $A = \bigoplus_{p \geq 0} A^p$

② for all $p, q \geq 0$, $\quad A^p A^q \subseteq A^{p+q}$

An element $x \in A^p$ is called a homogenous element of degree $p$.

eg. $k[X] = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} k \cdot X^i$, $\ k[X]/(X^2) = k \cdot 1 + kX$, $\ k[X_1, X_2, \dots, X_n]$

eg. Cohomology ring $H^{\bullet}(X, k)$, under cup product

<u>defn</u> If $A$ and $B$ are graded $k$-algebras, and $d \in \mathbb{Z}$, a <u>graded map of degree $d$</u> is a $k$-algebra map $f: A \to B$ such that $f(A^p) \subseteq B^{p+d}$ for all $p \geq 0$.

<u>note</u> a "graded map" or "grading-preserving map" means a degree $0$ map.

The main examples we want to discuss:

- Tensor algebras. $T(M)$
- Exterior algebras $\Lambda(M)$

# Tensor algebras of modules

<u>defn</u> Let $M \in k\text{-mod}$     eg. $V \in \mathrm{Vect}_{\mathbb{F}}$

The <u>tensor algebra of $M$</u> is the graded algebra over $k$ where

$$\left.\begin{aligned} T^0(M) &= k \\ T^1(M) &= M \\ T^2(M) &= M \otimes M \\ &\text{etc.} \end{aligned}\right\} \text{ ie. } T(M) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \left( \bigotimes^n M \right)$$

Multiplication is induced by

$$(\underbrace{a_1 \otimes a_2 \otimes \cdots \otimes a_k}_{\otimes^k M}, \underbrace{b_1 \otimes b_2 \otimes \cdots \otimes b_l}_{\otimes^l M}) \longmapsto \underbrace{\left(\bigotimes_{i=1}^k a_i\right) \otimes \left(\bigotimes_{j=1}^l b_j\right)}_{\bigotimes^{k+l} M}$$

<u>eg.</u> in the wild: Bar construction / Standard complex

$\rightsquigarrow$ projective ($\Rightarrow$ flat) resolution.

$\rightsquigarrow$ allow you to compute Hochschild homology, etc.

# Exterior Algebras of modules

<u>defn.</u> let $M \in k\text{-mod}$. Its <u>exterior algebra</u> is

$$\Lambda(M) = T(M)/J$$

where $J$ is the 2-sided ideal generated by all $m \otimes m$ for $m \in M$:

i.e. $J = \{ a \otimes m \otimes m \otimes b \,:\, a, b \in T(M),\ m \in M \}$.

The coset $(m_1 \otimes m_2 \otimes \cdots \otimes m_p) + J$ in $\Lambda(M)$ is denoted $m_1 \wedge m_2 \wedge \cdots \wedge m_p$ (wedge of $p$ factors)

<u>Fact.</u> $\Lambda(M) = k \oplus M \oplus \Lambda^2 M \oplus \Lambda^3 M \oplus \cdots$

<u>defn.</u> $\Lambda^p M$ is called the $p^{th}$ exterior power of $M$.

<u>defn.</u> A $k$-multilinear fn. $f : \prod_{i=1}^{p} M \to N$ is <u>alternating</u> if $f(m_1, \ldots, m_p) = 0$ whenever $m_i = m_j$ for some $i \neq j$.

## Universal Property

$$(m_1, \ldots, m_p) \longmapsto m_1 \wedge \cdots \wedge m_p$$

$$\begin{tikzcd}
\prod_{i=1}^{p} M \arrow[r, "h"] \arrow[dr, "f"'] & \Lambda^p M \arrow[d, dashed, "\tilde{f}"] \\
& N
\end{tikzcd}$$

alternating multilinear

Compare:

$$\begin{tikzcd}
\prod M \arrow[r, "h"] \arrow[dr, "f"'] & \otimes^p M \arrow[d, dashed, "\tilde{f}"] \\
& N
\end{tikzcd}$$

(multilinear)

theorem (Anticommutativity) let $x \in \Lambda^p M$, $y \in \Lambda^q M$.

Then $x \wedge y = (-1)^{pq}\, y \wedge x$.

pf. Use double induction.

p=0 or q=0: either $x$ or $y \in k$ ... Then $x \wedge y = y \wedge x$.

Base case: $p=1, q=1$

$$0 = (x+y)\wedge(x+y) = \cancel{x \wedge x} + x \wedge y + y \wedge x + \cancel{y \wedge y} \checkmark$$

(Continue w/ double induction).

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