MAT 250B HW02

[add your name here]

Due Friday, 1/19/24 at 11:59 pm on Gradescope

Reminder. *Your homework submission* ***must be typed*** *(TeX'ed) up in full sentences, with proper mathematical formatting.*

The following resources may be useful as you learn to use TeX and Overleaf:

- *Overleaf's introduction to LaTeX:*
  https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes
- *Detexify:*
  https://detexify.kirelabs.org/classify.html

Grading Most problems will be graded for completion out of 2 points: 1 if you wrote something meaningful, 2 if your solution is basically correct. A few select problems will be graded more carefully.

Solutions I will not be posting full solutions, but might post some abridged solutions for selected problems. You are responsible for looking at the solutions and determining whether your solutions are correct! This is a very important skill to have when you transition to doing research; you *must* learn how to check your own work. So, if you have any questions about whether your solution is valid, come ask me or the TA (after checking my abridged solutions).

Regarding figures You are not required to `tikz` your diagrams, but you're welcome to. Feel free to hand-draw your diagrams and insert them using a `figure` environment and/or

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\includegraphics[]{}
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Exercise 1

(Review)

(a) Let $M, N \in_R \mathbf{Mod}$, and let $f : M \to N$ be a module map. Prove that there is an exact sequence
$$0 \to \ker f \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0.$$

(b) Let $A \xrightarrow{f} B \to C \xrightarrow{h} D$ be an exact sequence. Prove that f is surjective if and only if h is injective.

Solution.

Exercise 2

(Review) A module $M \in_R \mathbf{Mod}$ is called **irreducible** if $M \neq 0$ and if 0 and M are the only submodules of M.

(a) Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as a generator.

(b) Determine all the irreducible $\mathbb{Z}$-modules.

Exercise 3

(a) Let k be a field, and let G be a **finite** group. Prove that $(kG)^{\mathrm{op}} \cong kG$.

(b) Let $\mathbb{H}$ be the division ring of real quaterions. Use part (a) to deduce that $\mathbb{H}^{\mathrm{op}} \cong \mathbb{H}$.

Solution.

Exercise 4

Prove that products (a) exist and (b) are unique up to unique isomorphism in ${}_R\mathbf{Mod}$.

Hint. *We covered the proof for coproducts in class, and verbally discussed the proof of uniqueness. Both these proofs are also in the book; use a similar diagram-chasing argument.*

Solution.

Exercise 5

What are the notions of (a) product and (b) coproduct in the category of groups **Grp**?

Hint. *Let $A, B \in$ **Grp** and think about different ways you know how to build a new (bigger) group from A and B. Once you've chosen your candidate constructions, prove that they satisfy the universal property defining products / coproducts.*

Solution.

Exercise 6

Let $\mathscr{C}$ be a category, let $A, B \in \mathscr{C}$ be objects, and let $f \in \mathrm{Hom}(A, B)$ be an **isomorphism**. Prove that its inverse $f^{-1} \in \mathrm{Hom}(B, A)$ is unique.

Solution.

Exercise 7

We now generalize products and coproducts to infinite families of objects. Let $\mathscr{C}$ be a category, and let $A_i \in \mathscr{C}$ for all $i \in I$, where I is an arbitrary indexing set.

Definition. The **coproduct** of the family of objects $\{A_i\}_{i\in I}$ is the pair $(C, \{\alpha_i : A_i \to C\}$ such that for any $X \in \mathscr{C}$ with and set of maps $\{f_i : A_i \to X\}$, there exists a unique morphism $\theta : C \to X$ such that the following diagram commutes for all i:

$$\begin{tikzcd} A_i \arrow[dr, "\alpha_i"] \arrow[ddr, bend right, "f_i"'] & \\ & C \arrow[d, dashed, "\exists!\ \theta"] \\ & X \end{tikzcd}$$

Such a coproduct, if it exists, is unique up to unique isomorphism, and is usually denoted $\coprod_{i\in I} A_i$. The **product** $P = \prod_{i\in I} A_i$ is also defined similarly:

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\ \theta"'] \arrow[ddr, bend left, "f_i"] & \\ P \arrow[dr, "\alpha_i"'] & \\ & A_i \end{tikzcd}$$

Once again, if the product exists, is unique up to unique isomorphism.

(a) In ${}_R\mathbf{Mod}$, the coproduct and product are denoted $\bigoplus_i A_i$ and $\prod_i A_i$. Prove that if the indexing set I is finite, then $\bigoplus_i A_i \cong \prod_i A_i$.

Hint. *First prove this for the family* $\{A, B\}$. *Then use induction.*

(b) Let **Ab** denote the category of $\mathbb{Z}$-modules (abelian groups). Prove that when $I = \mathbb{N}$, the coproduct and product are **not** in general isomorphic.

Hint. *Exhibit a counterexample. There's a hint in the lecture notes.*