# HW03

① $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ exact

$$\mathrm{Hom}_R(C,M) \xrightarrow{\beta^*} \mathrm{Hom}_R(B,M)$$
$$[f: C \to M] \longmapsto \beta^*(f) = [f\beta : B \xrightarrow{\beta} C \xrightarrow{f} M]$$

WTS $\beta^*$ is injective:

Suppose $\beta^*(f) = \beta^*(g)$, ie $f\beta = g\beta$. Let $c \in C$.

Choose any $b \in \beta^{-1}(c)$ (exists as $\beta$ surjective).

Then $f\beta(b) = g\beta(b) \Rightarrow f(c) = g(c)$. //

② $\boxed{\Rightarrow}$ Assume $P_1 \oplus P_2$ projective. Then $\exists$ free module $F$ and module $K$ where $F \cong K \oplus (P_1 \oplus P_2)$,

so both $P_1, P_2$ are also summands of $F$.

$\boxed{\Leftarrow}$ Assume $P_1, P_2$ projective, where

$F_1 \cong K_1 \oplus P_1$, $F_2 \cong K_2 \oplus P_2$ ($F_1, F_2$ free)

Then $P_1 \oplus P_2$ is a summand of the free module $F_1 \oplus F_2$. (generated by $B = B_1 \cup B_2$, where $B_i$ is a basis for $F_i$).

③ Recall that for modules, $Q_1 \oplus Q_2 \cong Q_1 \times Q_2$

and we can define projection to each coordinate

(even just on the underlying sets).

$$\begin{tikzcd}
Q_1 \arrow[dr, "\iota_1"] \arrow[rrr, bend left, "id_1"] & & & Q_1 \\
& Q_1 \oplus Q_2 \cong Q_1 \times Q_2 \arrow[urr, "\pi_1"] \arrow[drr, "\pi_2"] & & \\
Q_2 \arrow[ur, "\iota_2"'] \arrow[rrr, bend right, "id_2"'] & & & Q_2
\end{tikzcd}$$

$$a \longmapsto (a, 0)$$
$$b \longmapsto (0, b)$$

Below, let $0 \longrightarrow A \xrightarrow{\alpha} B$ be an exact sequence.

$\boxed{\Rightarrow}$ Assume $Q_1 \oplus Q_2$ is injective.

Given $\varphi : A \longrightarrow Q_1$,

form
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists \tilde{f}"] \\
& Q_1 \oplus Q_2 &
\end{tikzcd}$$
where $f = \varphi \oplus 0$,

ie $f(a) = \varphi(a) + 0$.

Then $\exists \tilde{f}$ lifting $f$. Then $\tilde{\varphi} = \pi_1 \tilde{f}$ lifts $\varphi$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f = \varphi \oplus 0"] \arrow[dd, bend right=60, "\varphi"'] & B \arrow[dl, "\tilde{f}"] \arrow[ddl, bend left=60, "\tilde{\varphi}"] \\
& Q_1 \oplus Q_2 \arrow[d, "\pi_1"] & \\
& Q_1 &
\end{tikzcd}$$

$\boxed{\Leftarrow}$ Assume $Q_1, Q_2$ are exact.

Given $\varphi : A \longrightarrow Q_1 \oplus Q_2$, consider the maps $(j = 1, 2)$

$$f_j : A \longrightarrow Q_j \quad \text{where} \quad f_j = \pi_j \varphi.$$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f_1"'] & B \arrow[dl, "\exists \tilde{f}_1"] \\ & Q_1 & \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f_2"'] & B \arrow[dl, "\exists \tilde{f}_2"] \\ & Q_2 & \end{tikzcd} \quad \Bigg\} \; Q_i \text{ are injective}$$

Let $\tilde{\varphi} = \tilde{f}_1 \oplus \tilde{f}_2 : B \longrightarrow Q_1 \oplus Q_2$

$$b \longmapsto f_1(b) + f_2(b)$$

Then check that $\tilde{\varphi}$ lifts $\varphi$:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\varphi = f_1 \oplus f_2"'] & B \arrow[dl, "\tilde{\varphi} = \tilde{f}_1 \oplus \tilde{f}_2"] \\ & Q_1 \oplus Q_2 & \end{tikzcd}$$

$$a \longmapsto f_1(a) + f_2(a)$$

④ Suppose (i) every module is projective.

Then given any SES

$$0 \to Q \to M \to N \to 0,$$

N is projective $\Rightarrow$ SES splits.

$\Rightarrow$ Q is injective.

Suppose (ii) every module is injective.

Then given any SES

$$0 \to L \to M \to P \to 0,$$

L is injective $\Rightarrow$ SES splits

$\Rightarrow$ P is projective.

(5) By classification of finitely generated $\mathbb{Z}$-mod,

$A \cong \mathbb{Z}/r_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/r_n\mathbb{Z}$. (may assume all $r_i > 1$)

(a) By Ex. 2, ISTS $\mathbb{Z}/r\mathbb{Z}$ is not projective.

$0 \to r\mathbb{Z} \to \mathbb{Z} \longrightarrow \mathbb{Z}/r\mathbb{Z} \to 0$ does not split.

because $r\mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z} \not\cong \mathbb{Z}$.

(b) By Ex 3, ISTS $\mathbb{Z}/r\mathbb{Z}$ is not injective.

Indeed, it is not divisible, since

$r \cdot (\mathbb{Z}/r\mathbb{Z}) = \{0\}$.

⑥

(a) We'll construct an isomorphism:

$$\prod_{i\in I} \mathrm{Hom}_R(B_i, A) \xrightarrow{\quad \Phi \quad} \mathrm{Hom}_R\Big(\bigoplus_{i\in I} B_i, A\Big)$$

$$f=(f_i)_{i\in I} \longmapsto \theta_f \qquad \text{given by univ. property.}$$

$$\begin{tikzcd}
B_j \arrow[dr, "\beta_j"] \arrow[ddr, bend right, "f_j"'] & & B_k \arrow[dl, "\beta_k"'] \arrow[ddl, bend left, "f_k"] \\
& \bigoplus_i B_i \arrow[d, "\exists!\theta"] & \\
& A &
\end{tikzcd}$$

($\theta$ is unique $\Rightarrow$ this is well-defined)

(By inspection, this respects "+" because if we are given $f=(f_i)$ and $g=(g_i)$, $\theta_f+\theta_g$ is a map that makes the corresponding diagram commute.)

Now define a candidate for the inverse map:

$$\mathrm{Hom}_R\Big(\bigoplus_{i\in I} B_i, A\Big) \xrightarrow{\quad \Psi \quad} \prod_{i\in I} \mathrm{Hom}_R(B_i, A)$$

$$\Big[F : \bigoplus_{i\in I} B_i \longrightarrow A\Big] \longmapsto (F\beta_i)_{i\in I}$$

$$\Big(B_i \xrightarrow{\beta_i} \bigoplus_{i\in I} B_i \xrightarrow{F} A\Big)$$

Finally, check that ① $\Psi\Phi = \mathrm{id}$ on $\prod \mathrm{Hom}_R(B_i, A)$

and ② $\Phi\Psi = \mathrm{id}$ on $\mathrm{Hom}_R(\bigoplus B_i, A)$

(b) Construct an isom:

$$\prod_{i \in I} \operatorname{Hom}_R(A, B_i) \xrightarrow{\quad \Phi \quad} \operatorname{Hom}_R\left(A, \prod_{i \in I} B_i\right)$$

$$f = (f_i)_{i \in I} \longmapsto \theta_f \quad \text{defined by U.P.}$$

again, this is well-defined because $\theta_f$ is unique

$$\begin{tikzcd}
 & A \arrow[ddl, bend right, "f_j"'] \arrow[d, "\exists! \theta"] \arrow[ddr, bend left, "f_k"] & \\
 & \prod_{i \in I} B_i \arrow[dl, "p_j"'] \arrow[dr, "p_k"] & \\
B_j & & B_k
\end{tikzcd}$$

Candidate inverse $\Psi$:

$$\operatorname{Hom}_R\left(A, \prod_{i \in I} B_i\right) \xrightarrow{\quad \Psi \quad} \prod_{i \in I} \operatorname{Hom}_R(A, B_i)$$

$$\left[F : A \longrightarrow \prod_{i \in I} B_i\right] \longmapsto (p_i F)_{i \in I}$$

$$\left(A \xrightarrow{F} \prod B_i \xrightarrow{p_i} B_i\right)$$

Again, check (by inspection) that these are abelian group homomorphisms and that $\Psi$ and $\Phi$ really are inverses.

⑦ (Exercise effectively removed from HW)

Let $F = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}$

$$F^* = \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_{n \in \mathbb{N}} \mathbb{Z}, \mathbb{Z}\right) \cong \prod_{n \in \mathbb{N}} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \prod_{n \in \mathbb{N}} \mathbb{Z}.$$

ISTS $\prod_{n \in \mathbb{N}} \mathbb{Z}$ is <u>not</u> free.

By way of contradiction, suppose $\prod_{n \in \mathbb{N}} \mathbb{Z}$ were free.

(Solution uses cardinality of $\bigoplus_{\mathbb{N}} \mathbb{Z}$ and $\prod_{\mathbb{N}} \mathbb{Z}$ and is quite involved)

⑧ The question asks when $\mathrm{Hom}_R(-, R)$ is an exact functor. This happens precisely when $R$ is an injective $R$-module!

(There are ways to characterize this, but none that I know of that only reference the ring structure of $R$.)