MAT 250B HW03

[add your name here]

Due Friday, 1/26/24 at 11:59 pm on Gradescope

Reminder Your homework submission **must be typed** (TeX'ed) up in full sentences, with proper mathematical formatting.

Grading Most problems will be graded for completion out of 2 points: 1 if you wrote something meaningful, 2 if your solution is basically correct. A few select problems will be graded more carefully.

Solutions I will not be posting full solutions, but might post some abridged solutions for selected problems. You are responsible for looking at the solutions and determining whether your solutions are correct! This is a very important skill to have when you transition to doing research; you *must* learn how to check your own work. So, if you have any questions about whether your solution is valid, come ask me or the TA (after checking my abridged solutions).

Regarding figures You are not required to `tikz` your diagrams, but you're welcome to. Feel free to hand-draw your diagrams and insert them using a `figure` environment and/or `includegraphics`.

Note If you are working on this HW before we've covered injective modules, you should start with Exercises 2, 5a, 6, and 7.

Exercise 1

Let $M \in {}_R\mathbf{Mod}$. Prove that the contravariant functor $\mathrm{Hom}_R(-, M)$ is left-exact.

Exercise 2

Let $P_1, P_2 \in {}_R\mathbf{Mod}$. Prove that $P_1 \oplus P_2$ is projective if and only if both P_1, P_2 are projective.

Exercise 3

Let $Q_1, Q_2 \in {}_R\mathbf{Mod}$. Prove that $Q_1 \oplus Q_2$ is injective if and only if both Q_1, Q_2 are injective.

Exercise 4

Prove that the following are equivalent for a ring R:

(i) Every R-module is projective.

(ii) Every R-module is injective.

Exercise 5

Let A be a nonzero finite abelian group.

(a) Prove that A is not a projective $\mathbb{Z}$-module.

(b) Prove that A is not an injective $\mathbb{Z}$-module.

Exercise 6

Let A be an R-module. Let I be a nonempty index set, and let $\{B_i\}_{i\in I}$ be a family of R-modules. Prove the following isomorphisms (of abelian groups):

(a) $\operatorname{Hom}_R(\bigoplus_{i\in I} B_i, A) \cong \prod_{i\in I} \operatorname{Hom}_R(B_i, A)$

(b) $\operatorname{Hom}_R(A, \prod_{i\in I} B_i) \cong \prod_{i\in I} \operatorname{Hom}_R(A, B_i)$

This is yet another way that the functors $\operatorname{Hom}_R(-, A)$ and $\operatorname{Hom}_R(A, -)$ behave differently.

Exercise 7

Let M be a left R-module. The **dual** (module) of M is $\operatorname{Hom}_R(M, R)$, which consists of R-module homomorphisms from M to the ground ring R. (Note that this is a generalization of the notation of a dual vector space $V^* = \operatorname{Hom}_{\mathbb{F}}(V, \mathbb{F})$, where $\mathbb{F}$ is a field.)

Show that the dual of the free $\mathbb{Z}$-module $\bigoplus_{i\in I} \mathbb{Z}$ is **not** free. *Hint: Use the previous exercise.*

Exercise 8

Under what conditions on R is "taking the dual" an exact functor? *Hint: Unpack the definitions. Once the definitions are clear, the solution will be quick.*