HW07

1

(i) $\Rightarrow$ (ii)

$\mu_n : A \longrightarrow A$ is a homomorphism.

A is tor-free $\Rightarrow$ (for $n \in \mathbb{N}$) $\mu_n(a) = na = 0$ iff $a = 0$ $\Rightarrow \mu_n$ injective.

$\operatorname{im}(\mu_n) = nA = A$ since A is divisble. $\Rightarrow \mu_n$ surjective.

$\Rightarrow \mu_n$ is an isom.

(ii) $\Rightarrow$ (iii)

Let μ_n^{-1} be the inverse of μ_n. (Note: $d \cdot \mu_d^{-1} = \mu_d \circ \mu_d^{-1} = id_A$).

Define the $\mathbb{Q}$-action as follows:

Let $q = \frac{m}{n} \in \mathbb{Q}$, $a \in A$. Let $q \cdot a = m\mu_n^{-1}(a)$.

$\hookrightarrow$ Well-defined? $\frac{m'}{n'} = \frac{m}{n}$, then WLOG $\frac{m'}{n'} = \frac{dm}{dn}$ for some $d \in \mathbb{N}$.

$$\frac{m'}{n'} \cdot a = \frac{dm}{dn} \cdot a = dm \cdot \mu_{dn}^{-1}(a) = md\mu_d^{-1}\mu_n^{-1}(a) = m\mu_n^{-1}(a) = \frac{m}{n} \cdot a$$

Check this action is $\mathbb{Q}$-linear.

(iii) $\Rightarrow$ (i)

Let A be a $\mathbb{Q}$-vector space.

- A is tor-free: BWOC suppose $\exists$ $a \neq 0$ in A and $n \in \mathbb{N}$ such that $na = 0$. But $\frac{1}{n} \cdot na = 1 \cdot a = a \neq 0$ ↯.

- A is divisible Let $n \in \mathbb{N}$, and $a \in \mathbb{N}$. Let $a' = \frac{1}{n} \cdot a$. Then $na' = a$. $\Rightarrow A = nA$.

② By Lecture 8, A embeds as a submodule of an injective $\mathbb{Z}$-mod D. View $A \subseteq D$. Then if $a \in A \cap T(D)$, $\exists$ some $n \in \mathbb{N}$ such that $na = 0$ Since A is tor-free, $a = 0$. $\Rightarrow A \cap T(D) = \{0\}$.

By an isomorphism theorem,

$$A \cong A/_{A \cap T(D)} \longrightarrow D/_{T(D)} \quad \text{is injective.}$$

$D/T(D)$ is divisible (quotient of injective $\mathbb{Z}$-mod) and torsion-free $\Longrightarrow$ by Ex 1, $D/T(D)$ is a $\mathbb{Q}$-VS.

③ Follows from $\left(\bigoplus_{i \in I} A_i\right) \otimes B \cong \bigoplus_{i \in I} (A_i \otimes B)$.

Need to check this commutes:

$$\begin{tikzcd}
\left(\bigoplus_{i \in I} A_i\right) \otimes B \arrow[r, "1 \otimes f"] \arrow[d, "\varphi_B", "\cong"'] & \left(\bigoplus_{i \in I} A_i\right) \otimes C \arrow[d, "\varphi_C", "\cong"'] \\
\bigoplus_{i \in I} (A_i \otimes B) \arrow[r, "\bigoplus 1 \otimes f"] & \bigoplus_{i \in I} (A_i \otimes C)
\end{tikzcd}$$

④

(a) Over any ring, free $\Rightarrow$ projective.

IRTS M projective $\Rightarrow$ M free.

But M projective $\Rightarrow \exists N$ s.t. $M \oplus N = F$ where F is free.

$\Rightarrow M$ is free by the prop.

(b)

- Let $S = \mathbb{Z} - \{0\}$. Then $\mathbb{Q} = S^{-1}\mathbb{Z}$ $(= \mathrm{Frac}(\mathbb{Z}))$.

  Localization is flat.

- BWOC suppose $\mathbb{Q}$ were projective.

  Then $\mathbb{Q}$ is a free $\mathbb{Z}$-module, so $\exists$ free basis $\mathcal{B} = \{b_i\}_{i \in I}$

  so that $\mathbb{Q} \cong \bigoplus_{i \in I} \mathbb{Z}$.

  Suppose there are at least two basis elements, say

  $b_1 = \frac{n_1}{d_1}$ and $b_2 = \frac{n_2}{d_2}$

  $$d_1 d_2 b_1 = d_1 d_2 \cdot \frac{n_1}{d_1} = d_2 n_1$$

  $$d_1 d_2 b_2 = d_1 d_2 \cdot \frac{n_2}{d_2} = d_1 n_2.$$

  Let $m = \mathrm{lcm}(d_2 n_1, d_1 n_2)$.

  ie $m = m_1 \cdot d_2 n_1 = m_2 \cdot d_1 n_2$.

  $\Rightarrow (m_1 d_1 d_2 \cdot b_1) - (m_2 d_1 d_2 \cdot b_2) = 0$

  $\Rightarrow \mathcal{B}$ is not a free basis.

  $\Rightarrow \mathbb{Q}$ must be rank 1. (It's not.) $\lightning$.