MAT 250B HW09

[add your name here]

Due Friday, 3/8/24 at 11:59 pm on Gradescope

Exercise 1

Determine whether the following polynomials are irreducible in the rings indicated. For those that are reducible, determine their factorization into irreducibles.

(a) $x^2 + x + 1$ in $\mathbb{F}_2[x]$

(b) $x^3 + x + 1$ in $\mathbb{F}_3[x]$

(c) $x^4 + 1$ in $\mathbb{F}_5[x]$

(d) $x^4 + 10x^2 + 1$ in $\mathbb{Z}[x]$

(e) $x^4 - 4x^3 + 6$ in $\mathbb{Z}[x]$

Exercise 2

Determine the degree of the following elements over $\mathbb{Q}$.

(a) $2 + \sqrt{3}$

(b) $1 + \sqrt[3]{2} + \sqrt[3]{4}$

(c) $\sqrt{3 + 2\sqrt{2}}$

(d) $\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}$

Exercise 3

Suppose $F = \mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_i^2 \in \mathbb{Q}$ for $i = 1, 2, \ldots, n$. Prove that $\sqrt[3]{2} \notin F$.

Exercise 4

Prove that $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Conclude that $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = 4$. Find the minimal polynomial of $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

Exercise 5

Determine the splitting field and its degree over $\mathbb{Q}$ for the following polynomials.

(a) $x^4 - 2$

(b) $x^4 + 2$

(c) $x^4 + x^2 + 1$

(d) $x^6 - 4$

Exercise 6

For any prime p and any nonzero $a \in \mathbb{F}_p$, prove that $x^p - x + a$ is irreducible and separable over $\mathbb{F}_p$. *Hint: Prove that if α is a root then $\alpha + 1$ is also a root.*

Exercise 7

Prove that $f(x)^p = f(x^p)$ for any polynomial $f(x) \in \mathbb{F}_p[x]$.