Midterm Exam Solutions

(1) We will show that S^1 is divisible.
Since we are writing S^1 multiplicatively, we need to check that $S^1 = (S^1)^n$ for $n \in \mathbb{Z}, n \neq 0$.

Consider the abelian group homomorphism

$$\begin{aligned} \varphi_n : S^1 &\longrightarrow S^1 \\ e^{i\theta} &\longmapsto e^{i\theta n} \end{aligned}$$

with image $(S^1)^n$.

This is an automorphism since it has an inverse

$$\begin{aligned} \psi_n : S^1 &\longrightarrow S^1 \\ e^{i\theta} &\longmapsto e^{i\theta/n} \end{aligned}$$

Therefore $S^1 = (S^1)^n$, so S^1 is divisible, and so S^1 is an injective $\mathbb{Z}$-module.

Alternate proof: Show $S^1 \cong \mathbb{R}/\mathbb{Z}$, show $\mathbb{R}$ is injective, and use the fact that quotients of injective modules over PIDs (such as $\mathbb{Z}$) are injective.

② (a) Let $\bigoplus_{i\in I} k$ and $\bigoplus_{j\in J} k$ be two free k-modules.

Using the lemma, then the recall, then the lemma,

$$\left(\bigoplus_{i\in I} k\right) \otimes \left(\bigoplus_{j\in J} k\right) \cong \bigoplus_{i\in I}\left(k \otimes \bigoplus_{j\in J} k\right)$$

$$\cong \bigoplus_{i\in I}\left(\left(\bigoplus_{j\in J} k\right) \otimes k\right)$$

$$\cong \bigoplus_{i\in I}\bigoplus_{j\in J} (k \otimes k).$$

Finally, $k\otimes k \cong k$, so this is the free module $\bigoplus_{\substack{i\in I,\\ j\in J}} k$.

(b) Since P_1, P_2 are projective, they are direct summands of free modules F_1 and F_2, i.e. there are modules Q_1 and Q_2, and indexing sets I and J such that

$$P_1 \oplus Q_1 \cong \bigoplus_{i\in I} k = F_1 \qquad \text{and} \qquad P_2 \oplus Q_2 \cong \bigoplus_{j\in J} k = F_2.$$

(For simplicity, we elide the isomorphisms and write $F_1 = P_1 \oplus Q_1$ and $F_2 = P_2 \oplus Q_2$.)

Now $F_1 \otimes F_2 \cong (P_1 \oplus Q_1) \otimes (P_2 \oplus Q_2)$

$$= P_1 \otimes P_2 \oplus P_1 \otimes Q_2 \oplus P_2 \otimes Q_1 \oplus P_2 \otimes Q_2$$

So $P_1 \otimes P_2$ is a summand of the free k-module

$$F_1 \otimes F_2 = \left(\bigoplus_{i\in I} k\right) \otimes \left(\bigoplus_{j\in J} k\right) = \bigoplus_{\substack{i\in I\\ j\in J}} k\otimes k \cong \bigoplus_{\substack{i\in I\\ j\in J}} k.$$

Therefore $P_1 \otimes P_2$ is also a projective k-module.

(3) Recall that $D \cong B \oplus C/_S$ where

$$S = \{ (f(a), -g(a)) \in B \oplus C \mid a \in A \}$$

By a unique isomorphism $\theta : D \to B \oplus C/_S$

consider the square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "\gamma' = \theta\gamma"] \\ B \arrow[r, "\beta' = \theta\beta"'] & B \times C/_S \end{tikzcd}$$

Suppose f is injective, and suppose $c \in C$ is in the kernel of γ'. Then $\gamma'(c) = 0 + S$

On the other hand, $\gamma'(c) = (0, c) + S$.

If $(0, c) \in S$, then $(0, c) = (f(a), -g(a))$ for some $a \in A$.

But if $f(a) = 0$, then $a = 0$, since f is injective.

Therefore $c = -g(a) = 0$ as well and so γ' is injective.

Since θ is an isomorphism, γ must also be injective.