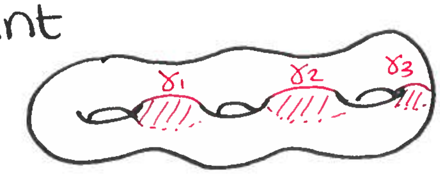
I. The Input Data

Defn: Let H be a genus-g handlebody. A set of attaching circles $\{\gamma_1, \dots, \gamma_g\}$ for H is a set of simple closed curves in $\Sigma = \partial H$ s.t.

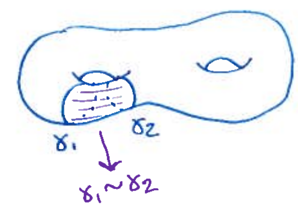
(1) the curves are pairwise disjoint

(2) $\Sigma - \gamma_1 - \cdots - \gamma_g$ is connected

(3) Each γ_i bounds a disk D_i in H.



↳ Note: (2) is equivalent to $\gamma_1, \dots, \gamma_g$ being linearly independent in homology



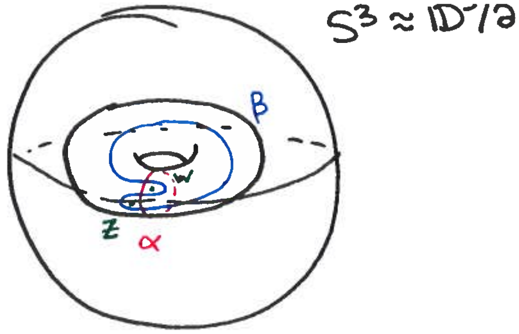
Defn: A doubly-pointed Heegaard diagram for a knot $K \hookrightarrow S^3$ is a tuple $(\Sigma_g, \alpha, \beta, w, z)$ consisting of

- Σ_g : the splitting surface of a Heegaard splitting $S^3 = H_\alpha \cup_{\Sigma_g} H_\beta$, oriented as ∂H_α.
- $\alpha = \{\alpha_1, \dots, \alpha_g\}$: attaching circles for H_α
- $\beta = \{\beta_1, \dots, \beta_g\}$: attaching circles for H_β
- w, z : basepoints in $\Sigma - \alpha - \beta$

such that

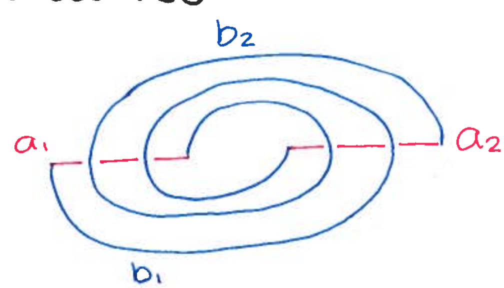
- $K \cap \Sigma_g = \{\overset{(+)}{w}, \overset{(-)}{z}\}$ $\qquad f: X \to Y$, $f^{-1}(\partial Y) = \partial X$
- $K \cap H_\alpha$: a properly embedded arc in $H_\alpha \setminus \amalg D_i^\alpha$
- $K \cap H_\beta$: a properly embedded arc in $H_\beta \setminus \amalg D_i^\beta$

Ex:



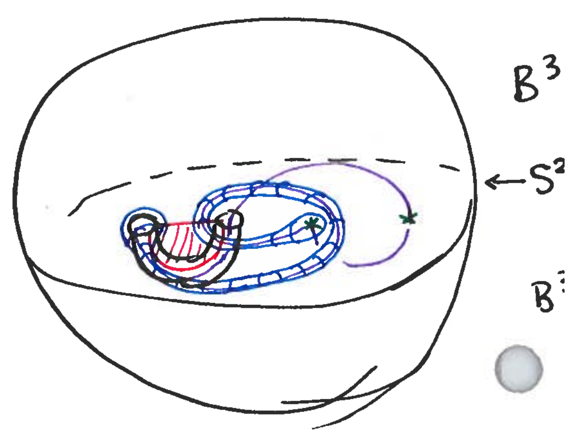
- $K \cap \Sigma_g$
- $K \cap H_\alpha$
- $K \cap H_\beta$

Defn: A <u>bridge presentation</u> of a knot consists of segments $a_1, \dots, a_{g+1}, b_1, \dots, b_{g+1}$ s.t.
(1) the a curves are disjoint
(2) the b curves are disjoint
(3) the b curves pass over the a curves



Construction:
1. Start w/ a bridge presentation of K
2. Attach a 1-handle to the ends of each $a_1, \dots, a_g$
3. Let $\{w, z\} = \partial a_{g+1}$, ~~[illegible]~~
4. Let α_i be 

5. Let β_i be the boundary of a regular nbhd of b_i $(i \neq g+1)$



Side Quest: Pseudoholomorphic Curves

Defn: An almost complex structure J on a smooth, $2n$-dim'l mfd M is a smooth assignment of linear maps $J_p : T_pM \to T_pM$ s.t. $J_p \circ J_p = -Id$.

smooth as a map $J: TM \to TM$

On each tangent space, J gives the $\mathbb{C}$-action

$$(a+bi)v = av + bJ_p(v),$$

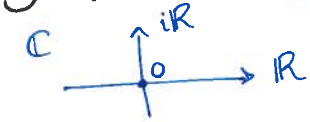
which turns T_pM into an n-dim'l v.s. over $\mathbb{C}$.

Ex: Every complex manifold has a canonical almost complex structure

↳ ~~Locally,~~ If $\varphi : \mathbb{C}^n \to M$ is a chart w/ coords $(x_1+iy_1, \ldots, x_n+iy_n)$, then $\psi : \mathbb{R}^{2n} \to M$ is a chart w/ coords $(x_1, y_1, \ldots, x_n, y_n)$, and $J_p = \begin{pmatrix} & -Id \\ Id & \end{pmatrix}$ in the basis $(\partial/\partial x_1, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n})$

Defn: A submfd L of a complex mfd w/ almost cx. str. J is totally real if $T_pL \cap J(T_pL) = \{0\}$

↳ analogy:



Defn: A smooth map $\varphi : \underbrace{(\Sigma, j)}_{\text{Riemann surface}} \to \underbrace{(M, J)}_{\text{almost cx. mfd}}$ is a pseudoholomorphic curve if $d\varphi$ is $\mathbb{C}$-linear, i.e., if $J \circ d\varphi = d\varphi \circ j$.

↳ simply put, they are the correct notion of a curve on an almost cx mfd.

II. Auxiliary Objects $\square$ = technical condition

<u>Defn</u>: The <u>g-fold symmetric product</u> of a smooth mfd M is

$$\mathrm{Sym}^g(M) = M^{\times g} / S_g,$$

where S_g acts by permuting the factors of $M^{\times g}$.

↳ $M^{\times g}$ is <u>ordered</u> tuples of g points in M
$\mathrm{Sym}^g(\Sigma)$ is <u>unordered</u> ~~tuples~~ sets of g points in M

[1] Given $(\Sigma_g, \alpha, \beta, w, z)$, $\mathrm{Sym}^g(\Sigma)$ is a smooth $2g$-dim'l manifold $\boxed{\text{with cx. str.}}$ induced by the cx. str. on Σ.

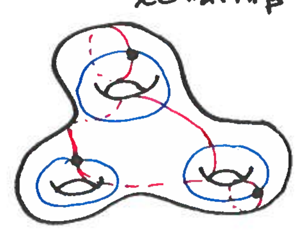
[2] $\mathrm{Sym}^g(\Sigma)$ contains the smoothly-embedded g-tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g / S_g$$

$$\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g / S_g$$

which are $\boxed{\text{totally real}}$ and can be arranged transversally so that $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is finite.

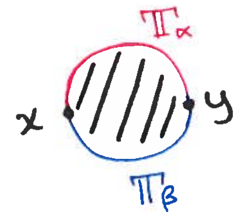
$x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$



↳ How to see them: A point in...
- $\mathbb{T}_\alpha$ is a choice of one point per α_i
- $\mathbb{T}_\beta$ is a choice of one point per β_j
- $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is a choice of intersections $\alpha_i \cap \beta_j$ using every curve exactly once.

Let $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

<u>Defn</u>: A <u>Whitney disk</u> connecting x to y is a cts. map $u : \mathbb{D}^2 \subseteq \mathbb{C} \to \mathrm{Sym}^g(\Sigma)$ s.t. u sends



(*)

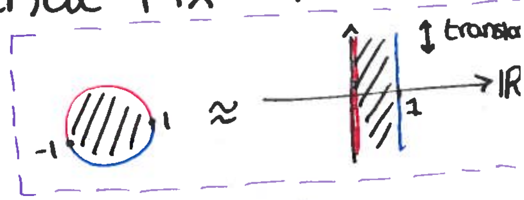
[3] Let $\pi_2(x,y)$ be the htpy classes rel (*) of Whitney disks connecting x to y.

[4] For $\varphi \in \pi_2(x,y)$, let $\mathcal{M}(\varphi)$ be the moduli space of pseudoholomorphic representatives of φ

↳ Generic perturbations of the α curves make $\boxed{\mathcal{M}(\varphi) \text{ a smooth mfd}}$

↳ $\dim(\mathcal{M}(\varphi)) = \mu(\varphi)$ — the Maslov index

To count disks uniquely, we mod out by automorphisms of $\mathbb{D}^2 \subseteq \mathbb{C}$ that fix -1 and 1:

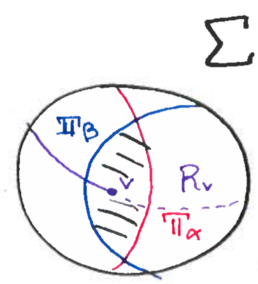


$$\widehat{\mathcal{M}}(\varphi) = \mathcal{M}(\varphi)/\mathbb{R}$$

(unparametrized) disks — parametrized disks

↳ Smooth mfd of dimn $\mu(\varphi) - 1$

↳ Finite when 0-dim'l



[5] Let $v \in W \cup Z$, $\varphi \in \pi_2(x,y)$. Define

$$R_v = \{v\} \times \mathrm{Sym}^{g-1}(\Sigma)$$

$$n_v(\varphi) = i(\varphi, R_v)$$

<u>Fact</u>: A generic pseudoholo. rep of φ does not intersect R_v if $n_v(\varphi) = 0$. i.e. φ realizes the alg. intersection # geometrical

III. The Knot Floer Complex

Given a doubly-pointed Heegaard diagram $\mathcal{H} = (\Sigma_g, \alpha, \beta, w, z)$ for $K \hookrightarrow S^3$, the complex $\widehat{gCFK}(\mathcal{H})$ is defined by

(1) <u>Generators</u>: Intersection pts in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$

(2) <u>Differential</u>:

# of unparametrized pseudoholo. disks

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi) = 1 \\ n_z(\varphi) = n_w(\varphi) = 0}} (\# \widehat{\mathcal{M}}(\varphi))\, y$$

$\mu(\varphi) = 1 \Rightarrow |\widehat{\mathcal{M}}(\varphi)| < \infty$

$n_z(\varphi) = n_w(\varphi) = 0 \Rightarrow \varphi$ doesn't pass over z, w

(3) <u>Bigrading</u> (M, A):

- Maslov grading (homological): Determined (up to a constant) by

$$M(x) - M(y) = \mu(\varphi)$$

for any $\varphi \in \pi_2(x,y)$

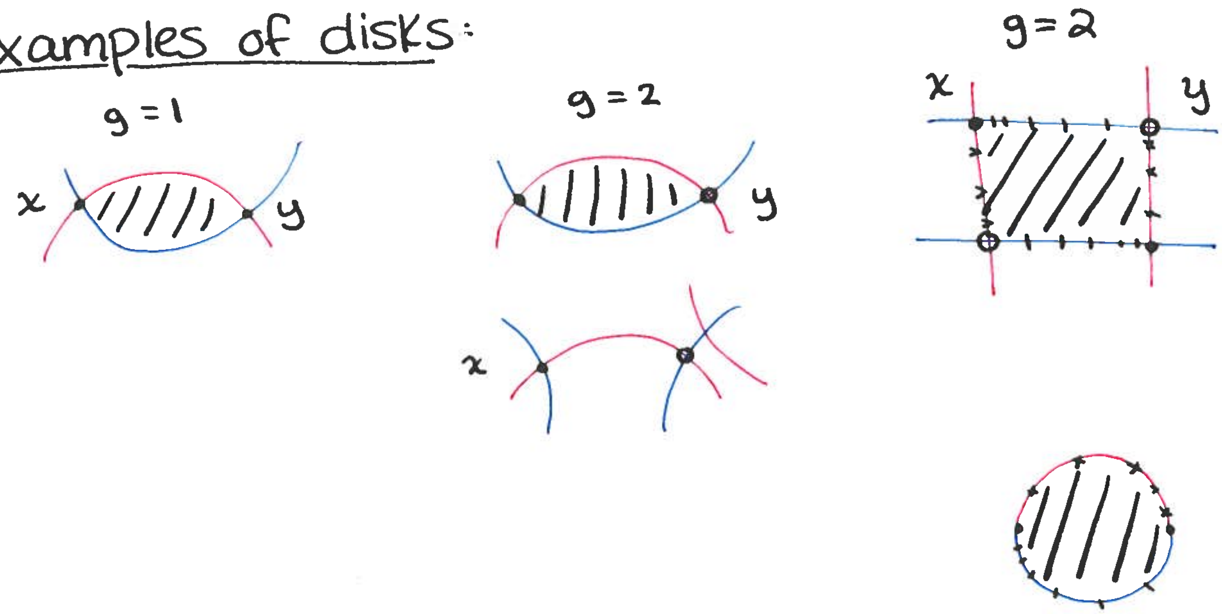
independent of φ

- Alexander grading (knot): Determined by
  - (i) $A(x) = A(y)$ when $\pi_2(x,y) \neq \emptyset$
  - (ii) $\displaystyle\sum_{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} (-1)^{M(x)} q^{A(x)} = \Delta_K(q)$

<u>Thm</u> [Oz-Sz, Ras.]: The homology $\widehat{HFK}(\mathcal{H})$ of $\widehat{gCFK}(\mathcal{H})$ is independent of $\mathcal{H}$, thus giving the invariant $\widehat{HFK}(K)$.

IV. Computations.

<u>Examples of disks</u>:



<u>Defn</u>: A Heegaard diagram is <u>nice</u> if any region that does not contain a w_i is a square or a bigon

<u>Thm</u> [Sarkar-Wang '10]:

(1) $\widehat{HFK}(K)$ can be computed combinatorially from a nice Heegaard diagram

(2) Gave an algorithm to convert any pointed Heegaard diagram into a nice diagram.

- Simultaneously, Manolescu-Ozsvath-Sarkar computed $\widehat{HFK}(K)$ combinatorially from grid diagrams

- Sarkar-Wang uses Lipschitz's cylindrical reformulation, which replaces $\mathrm{Sym}^g(\Sigma)$ with $\Sigma \times [0,1] \times \mathbb{R}$.