Hegaard Floer – Week 4

10/21/25
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I. Wrapping up HFK

Previously, we defined

- $g\widetilde{CFK}$, $g\widehat{CFK}$, $gCFK^-$
- A filtration on a chain complex.

We now define new chain complexes $\widetilde{CFK}$, $\widehat{CFK}$, and CFK^- without z-basept information (and thus w/o knot information) and then recover their "g" counterparts via a filtration.

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(g)CFK flavors

	$g\widetilde{CFK}$	$g\widehat{CFK}$	$gCFK^-$
$\mathcal{H}$	$(\Sigma_g, \alpha, \beta, \vec{w}, \vec{z})$	$(\Sigma_g, \alpha, \beta, w, z)$	$(\Sigma_g, \alpha, \beta, \vec{w}, \vec{z})$
ring	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}[\mathcal{U}_1, \dots, \mathcal{U}_k]$
gen	$\mathbb{T}_\alpha \cap \mathbb{T}_\beta$	$\mathbb{T}_\alpha \cap \mathbb{T}_\beta$	$\mathbb{T}_\alpha \cap \mathbb{T}_\beta$
$\partial(x)$	$\sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi)=1 \\ n_{w_i}(\varphi)=0 \\ n_{z_i}(\varphi)=0}} \# \hat{M}(\varphi) y$	$\sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi)=1 \\ n_w(\varphi)=0 \\ n_z(\varphi)=0}} \# \hat{M}(\varphi) y$	$\sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi)=1 \\ n_{z_i}(\varphi)=0}} \# \hat{M}(\varphi) \cdot \mathcal{U}_1^{n_{w_1}} \cdots \mathcal{U}_k^{n_{w_k}} y$
gr	M, A	M, A	M, A

only applies to gCFK :)

To recover gCFK from CFK, we'll need some more filtration infrastructure first:

<u>Defn:</u> Let $\{F_nA\}$ be a filtration of a module A. Its n^{th} <u>associated graded module</u> G_nA is

$$G_nA = F_nA / F_{n-1}A.$$

Consider the filtered chain cx (F_nC_*, ∂):

$$\begin{tikzcd}[row sep=small]
& \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] & C_n \arrow[r, "\partial"] & C_{n-1} \arrow[r] & \cdots \\
G_{k+1}C_{n+1} & & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \\
& & F_kC_{n+1} \arrow[r, "\partial"] & F_kC_n \arrow[r, "\partial"] & F_kC_{n-1} & \\
G_kC_{n+1} & & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \\
& & F_{k-1}C_{n+1} \arrow[r, "\partial"] & F_{k-1}C_n \arrow[r, "\partial"] & F_{k-1}C_{n-1} & \\
G_{k-1}C_{n+1} & & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & \\
& & F_{k-2}C_{n+1} \arrow[r, "\partial"] & F_{k-2}C_n \arrow[r, "\partial"] & F_{k-2}C_{n-1} &
\end{tikzcd}$$

∂ induces the boundary map $\partial: G_kC_i \to G_kC_{i-1}$, which yields the <u>associated graded chain complexes</u> $\{G_kC_*\}$:

$$\cdots \longrightarrow G_{k+1}C_{n+1} \xrightarrow{\partial} G_{k+1}C_n \xrightarrow{\partial} G_{k+1}C_{n-1} \longrightarrow \cdots$$

$$\cdots \longrightarrow G_kC_{n+1} \xrightarrow{\partial} G_kC_n \xrightarrow{\partial} G_kC_{n-1} \longrightarrow \cdots$$

$$\cdots \longrightarrow G_{k-1}C_{n+1} \xrightarrow{\partial} G_{k-1}C_n \xrightarrow{\partial} G_{k-1}C_{n-1} \longrightarrow \cdots$$

Defn: For $K \hookrightarrow S^3$, the Knot filtration $\{F(K,n)\}$ on CFK° is defined by

$$F(K,i) = \{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid A(x) \leq i\}.$$

↳ That is,

$$\cdots \subseteq F(K,i-1) \subseteq F(K,i) \subseteq F(K,i+1) \subseteq \cdots$$

The associated graded modules are

$$F(K,i)/F(K,i-1) \cong \{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid A(x) = i\},$$

which precisely recover the Alexander grading! Therefore, the associated graded cx of CFK equipped w/ the knot filtration is iso. to $gCFK$.

Rmk: Since the CFK do not encode the knot, they are actually chain cx's for $HF(S^3)$:

$$\widetilde{CFK}(\mathcal{H}) \overset{H_*}{\rightsquigarrow} H_*(T^{k-1})$$

$$\widehat{CFK}(\mathcal{H}) = \widehat{CF}(S^3) \rightsquigarrow \widehat{HF}(S^3)$$

$$CFK^-(\mathcal{H}) = CF^-(S^3) \rightsquigarrow HF^-(S^3)$$

It follows that since $\widehat{HFK}(\mathcal{H}) \cong H_*(T^{k-1})$ and

$$H_n(T^{k-1}) \cong \begin{cases} \mathbb{Z} & n = k-1 \\ \mathbb{Z}^{\binom{?}{k}} & n < k-1, \end{cases}$$

we can give $\widehat{HFK}(\mathcal{H})$ an absolute Maslov grading by fixing a generator

$$\varphi \in H_{k-1}(T^{k-1}) \cong \mathbb{Z}$$

to have Maslov grading 0 in $\widehat{HFK}(\mathcal{H})$.

Finally, to finish out HFK defns, we ought to at least acknowledge that HFL exists:

<u>Rmk</u> (Manolescu): For a link $L \hookrightarrow S^3$ with ℓ components, HFK generalizes to HFL with

- $\geq 2\ell$ basepts
- ℓ Alexander gradings
- $\widehat{HFL}(L)$ is $(\ell+1)$-graded over $\mathbb{Z}$
- $HFL^-(L)$ is multi-graded over $\mathbb{Z}[U_1, \ldots, U_k]$.

See below for a summary of HFK. Not pictured is $CFK^-(K)$, which is $CFL^-(\mathcal{H})$ with the knot filtration. It is an invariant up to filtered chain htpy.

HFK flavors

filter & take assoc. gr.

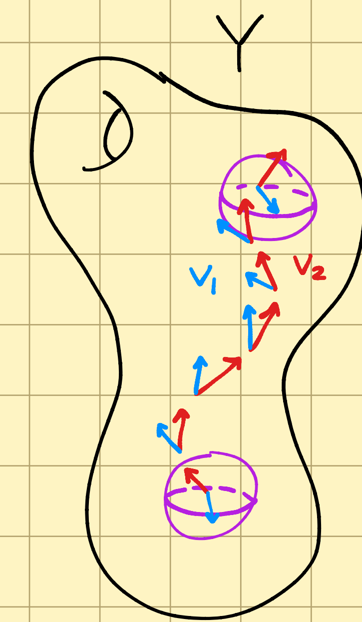
Chain cx	Ring	Basepts	Homology
$g\widetilde{CFK}(\mathcal{H})$	$\mathbb{Z}$	$\vec{w}, \vec{z}$ no crossings	$\widetilde{HFK}(\mathcal{H})$
↓ specializes to			
$g\widehat{CFK}(\mathcal{H})$ up to chain htpy	$\mathbb{Z}$	w, z no crossings	$\widehat{HFK}(K)$ up to iso.
$gCFK^-(\mathcal{H})$ up to chain htpy	$\mathbb{Z}[U_1, \dots, U_k]$	$\vec{w}, \vec{z}$ crosses $\vec{w}$	$HFK^-(K)$ up to iso.
$\widetilde{CFK}(\mathcal{H})$	$\mathbb{Z}$	$\vec{w}$ no crossings	$H_*(T^{k-1})$
↓ specializes to			
$\widehat{CFK}(\mathcal{H})$	$\mathbb{Z}$	w no crossings	$\widehat{HF}(S^3) \cong \mathbb{Z}$
$CFK^-(\mathcal{H})$	$\mathbb{Z}[U_1, \dots, U_k]$	$\vec{w}$ crosses $\vec{w}$	$HF^-(S^3) \cong \mathbb{Z}[U]$
$CFK^\infty(K)$ up to filtered chain htpy	$\mathbb{Z}[U_1, \dots, U_k, U_1^{-1}, \dots, U_k^{-1}]$	$\vec{w}$ crosses $\vec{w}$	recovers • $CFK^{+,-,\wedge}$ • $\widehat{CF}(S^3)$ ↑ obviously, not homologies

Knot invariant

Side Quest: Spin^c Structures

<u>Defn</u> [Turaev]: The set of <u>spin^c structures</u> spin^c(Y) of a 3-mfd Y is

$$\text{spin}^c(Y) = \{ v \in \mathfrak{X}(Y) \mid \|v\| = 1 \} / \sim,$$



where

$$v_1 \sim v_2 \iff v_1\big|_{Y \setminus B} \simeq v_2\big|_{Y \setminus B}.$$

for some collection B of open 3-balls

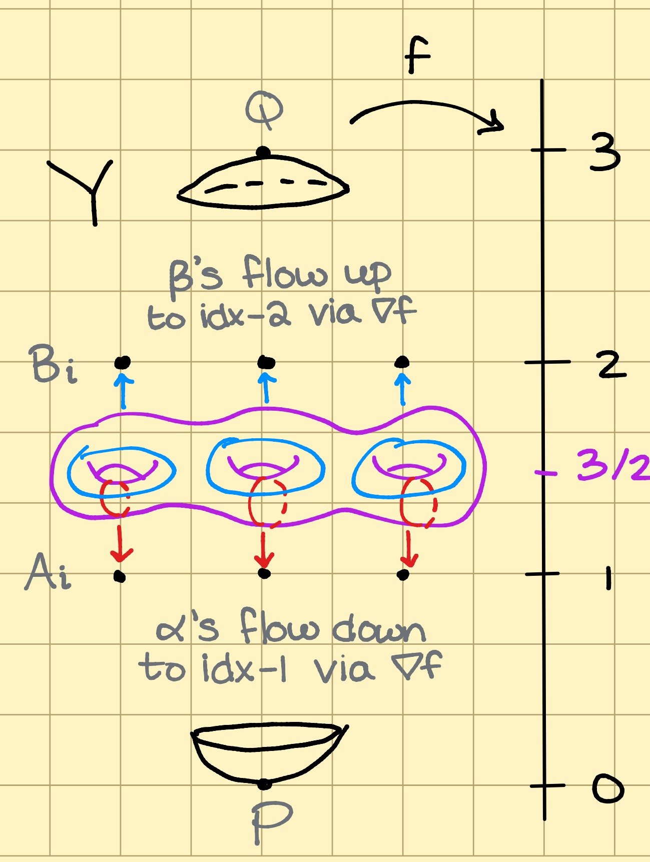
↳ The real defn is in terms of lifts of the frame bundle of Y. This is just one of many equivalent defns when restricted to 3-mfds. Turaev instead calls these "Euler structures".

⚠ vibes ahead

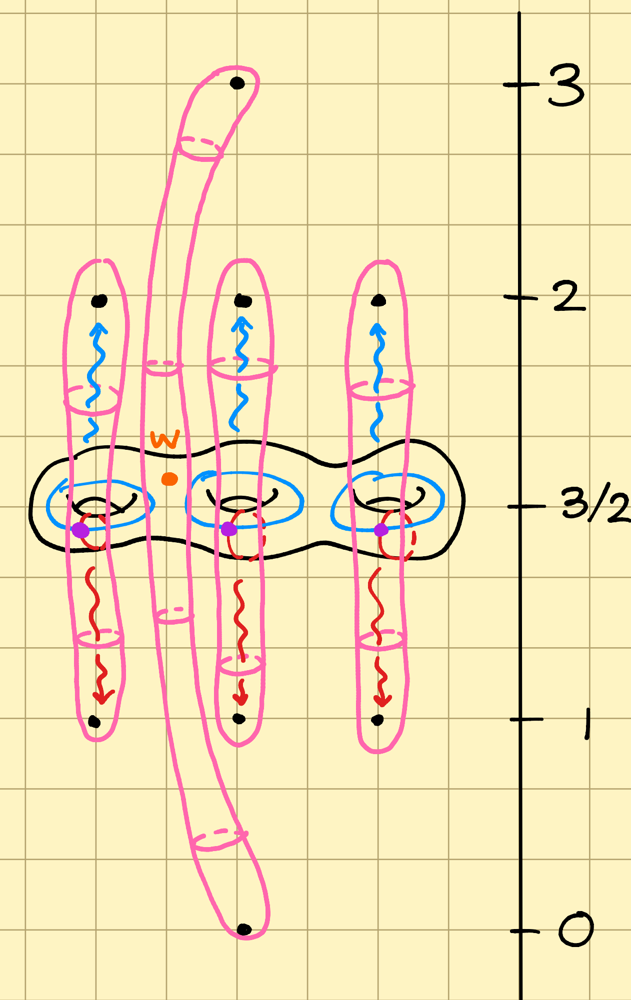
Given a basept w and a generator x in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we associate a spin^c structure <u>$s_w(x)$</u> $\in$ spin^c(Y) as follows:

1. Let $\mathcal{H}$ be given by a self-indexing Morse function $f: Y \to \mathbb{R}$ having

    - one idx-0 crit. pt. P
    - g idx-1 crit. pts A_i
    - g idx-2 crit. pts B_i
    - one idx-3 crit. pt. Q



2. Let $x = \{x_1, \ldots, x_g\} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.
Each x_i flows down to A_i and up to B_i.



tubular nbhd

3. Cut out a worm around the trajectory of each x_i.

4. Cut out a worm around the trajectory from Q to P going through w.

5. $\mathrm{idx}(\nabla f)$ on ∂worm is 0 since each worm has crit. pts of opposite parities.

G&P 3.6 exc. 18

6. Extend ∇f from $Y \setminus$ worms to a non-vanishing v.f. on Y.

G&P 3.6 Extension Thm

7. $Sw(x) = \left[\dfrac{v}{\|v\|} \right]$

<u>Summary</u>: Cut out the bits where ∇f vanishes (the crit. pts), then extend to a non-vanishing v.f.

III. HF(Y)

Let Y be a 3-mfd with $b_1(Y) = \text{rk}\, H_1(Y;\mathbb{Z}) = 0$.

↳ $b_1(Y) > 0$ requires an extra admissibility condition on $\mathcal{H}$ to ensure that $\widehat{\mathcal{M}}(\varphi)$ is finite. The condition is different for each flavor.

Our input data is

- a pointed Heegaard diagram $\mathcal{H} = (\Sigma_g, \alpha, \beta, w)$
- a spin^c-structure s
- $\boxed{\text{auxiliary data}}$ ← it's a lot to say, and our invariants will turn out to be independent of such choices

From this data, we build 4 chain cx's whose homologies are invariants of (Y, s): ← see next page

(1) $\widehat{CF}(\mathcal{H}, s) \overset{H_*}{\rightsquigarrow} \widehat{HF}(Y, s)$

(2) $CF^\infty(\mathcal{H}, s) \rightsquigarrow HF^\infty(Y, s)$

(3) $CF^-(\mathcal{H}, s) \rightsquigarrow HF^-(Y, s)$

(4) $CF^+(\mathcal{H}, s) \rightsquigarrow HF^+(Y, s)$.

By summing over all spin^c structures, we obtain invariants of just Y:

$$HF^{\circ}(Y) = \bigoplus_{s \in \text{spin}^c(Y)} HF^{\circ}(Y, s).$$

CF flavors	(1) $\widehat{CF}(\mathcal{H},s)$	(2) $CF^\infty(\mathcal{H},s)$
$\mathcal{H}$	$(\Sigma_g, \alpha, \beta, w)$	$(\Sigma_g, \alpha, \beta, w)$
ring	$\mathbb{Z}$	$\mathbb{Z}$
gen	$\mathcal{G} = \{x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid s_w(x) = s\}$	$\{[x,i] \mid x \in \mathcal{G},\ i \in \mathbb{Z}\}$
∂	$\partial(x) = \sum\limits_{y \in \mathcal{G}} \sum\limits_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi)=1 \\ n_w(\varphi)=0}} \#\hat{M}(\varphi)\, y$	$\partial([x,i]) = \sum\limits_{y \in \mathcal{G}} \sum\limits_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi)=1 \\ n_w(\varphi)=0}} \#\hat{M}(\varphi)[y, i - n_w(\varphi)]$
gr	$gr(x) - gr(y) = \mu(\varphi) - 2n_w(\varphi)$	$gr([x,i]) - gr([y,j]) = gr(x) - gr(y) + 2i - 2j$

(3) $CF^-(\mathcal{H},s) \subseteq CF^\infty(\mathcal{H},s)$, gen'd by pairs $[x,i]$, $i<0$

(4) $CF^+(\mathcal{H},s) = CF^\infty(\mathcal{H},s) / CF^-(\mathcal{H},s)$