LECTURE 1

* See Canvas announcement for ~~office~~ hours survey.

Alg. Topology often only sees spaces up to "homotopy".

"continuous deformation".

eg $A \simeq \mathbb{A} \simeq \mathbb{P} \simeq P \simeq O \simeq e$ etc. But $\not\simeq X \simeq Y \simeq Z \simeq \bullet$

"Homotopy equivalence" ($\simeq$) will be an equivalence relation on topological spaces.

First consider a concrete deformation from a space X to a subspace $A \subset X$.

This is a special case of a "homotopy":

<u>defn</u> A <u>deformation retraction</u> of a space X onto a subspace A is a continuous family of continuous maps:

$$\{ f_t : X \to X \}_{t \in [0,1]}$$

such that

① $f_0 = \mathbb{1}$ ← identity map id_X ② $f_1(X) = A$ ③ $f_t|_A = \mathbb{1}$ ← id_A $\forall t$.

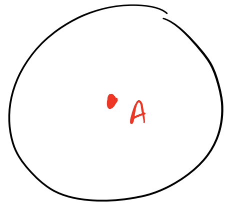
<u>Continuity</u>:

$$F : X \times \underset{\text{"}I\text{"}}{[0,1]} \longrightarrow X \qquad (x,t) \longmapsto f_t(x)$$

must be continuous.

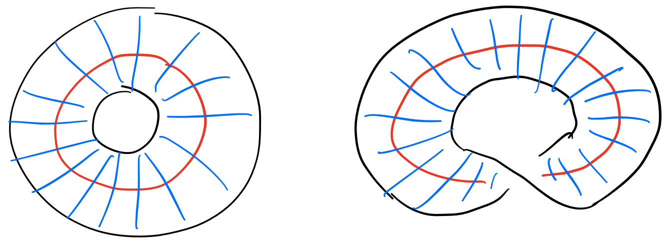
<u>Rmk</u> Assume everything is continuous. That's the appropriate structure of morphisms in the cat. of top spaces.

e.g.



$X = \mathbb{D}^2$ unit disk in the $\mathbb{R}^2$ plane

$f_t(x) = (1-t)x$



Annulus Möbius band

* see book for more pictures.

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defn. A <u>homotopy</u> is a family of maps $\{f_t : X \to Y\}_{t \in [0,1]}$ such that the associated map

$$\begin{aligned} F : X \times I &\longrightarrow Y \\ (x,t) &\longmapsto f_t(x) \end{aligned}$$

is continuous.

We say two maps $f_0, f_1 : X \to Y$ are <u>homotopic</u> if there exists a homotopy f_t relating them.

In this case we write $f_0 \simeq f_1$

e.g.

$$\begin{aligned} i : \mathbb{D}^2 &\longrightarrow \mathbb{R}^2 \\ p &\longmapsto p \end{aligned} \qquad \begin{aligned} f : \mathbb{D}^2 &\longrightarrow \mathbb{R}^2 \\ p &\longmapsto p + (1,0) \end{aligned} \qquad \begin{aligned} g : \mathbb{D}^2 &\longrightarrow \mathbb{R}^2 \\ p &\longmapsto 2p \end{aligned}$$

are all homotopic maps: $i \simeq f \simeq g$.

$\mathbb{D}^n$, S^n, $\mathbb{R}^n$, $*$, $\mathbb{RP}^n$ when discuss cell cpx. Euler char.

projection, idempotent in algebra, geometry: $p^2=p$.

Alternatively: $p: X \to X$ where ① $p(X)=A$ ② $p(a)=a$ $a\in A$

In topology, the appropriate analogue is a "retraction":

<u>defn</u> Let $A \subset X$. subspace.

A map $r: X \to X$ is a <u>retraction</u> if

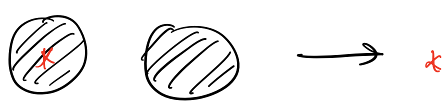
① $r(X)=A$ and ② $r|_A = \mathbb{1}$.

Equivalently, $r: X \to X$ is a retraction if $r^2 = r$.

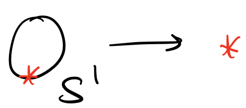
an example of a

(ex) <u>Rmk.</u> A deformation retraction of X onto $A \subset X$ is a homotopy from id_X to a retraction of X onto A. (think time)

<u>Rmk</u> Not all retractions are f_1 of a deformation retraction $\{f_t\}$.

eg 1  is a retraction.

But X is not path cntd, so you can't have a cts family of maps for a deformation retraction.

<u>eg 2</u> 

Also cannot deformation retract a circle onto a point.

Key: $S^1 \not\simeq pt$ These spaces are not "homotopy equivalent"...

Observe

If a space X deformation retracts onto $A \subset X$ via $\{f_t : X \to X\}$,

and $\rho : X \to A$ is the assoc'd retraction

$\iota : A \to X$ is the inclusion

then we have $\rho\iota = \mathbb{1}$ and $\iota\rho \simeq \mathbb{1}$

homotopic! not necessarily equal!

$A \overset{\iota}{\hookrightarrow} X \overset{\rho}{\twoheadrightarrow} A$

$\underbrace{X \overset{\rho}{\twoheadrightarrow} A \overset{\iota}{\hookrightarrow} X}$

= "r" from before, where $r^2 = r$

The htpy from $\mathbb{1}_X$ to $r = \iota\rho$ is $\{f_t\}$.

Generalizing:

defn. A map $f : X \to Y$ is a <u>homotopy equivalence</u> if there exists a map $g : Y \to X$ s.t.

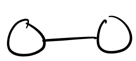
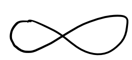
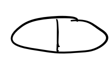
$fg \simeq \mathbb{1}$ and $gf \simeq \mathbb{1}$.

Then X is <u>homotopy equivalent</u> to Y,

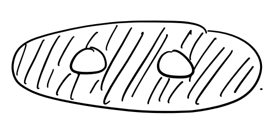
ie X and Y have the same <u>homotopy type</u>

← ie equivalence class

and we may write $\underline{X \simeq Y}$.

eg. In book: , ,  are all htpy equiv,

but none of them is a def. retract of any other.

(They are all def. retracts of )

LECTURE 2

Office Hours set: M 4-5 F 2-3

Recall $f \simeq g$, $X \simeq Y$ meanings.

defn. A space X is called contractible if it has the homotopy type of a point $*$.

- This is equivalent to saying that $\overbrace{id_X \simeq c}^{id_X \text{ is "null homotopic"}}$ where c is a constant map $c: X \to X$, $x \mapsto p \leftarrow$ fixed point

Let's unpack this. By definition.

X and $*$ are homotopy equivalent

$\Leftrightarrow \exists$ $f: X \to *$ (constant map) and $g: * \to X$

s.t.

$$\underbrace{fg}_{\text{(for free)}} \simeq id_* \quad \text{and} \quad \underbrace{gf} \simeq id_X$$

$c = g \circ f : X \to * \to X$ is a constant map; $p = g(*)$.

- ⚠ This is however in general weaker than saying the space deformation retracts to a point!

Indeed c is a retraction, and there is a htpy $\mathbb{1}_X \simeq c$. However this htpy need not be a deformation retraction because there may be some $t \in [0,1]$ where $f_t(p) \neq p$!

ex. In book, Chp 0 ex. 5–6 show an example.
ex 4 gives a more lenient defn:

<u>defn.</u> A <u>deformation retraction in the weak sense</u> of a space X to a subspace $A \subset X$ is a htpy $\{f_t : X \to X\}$ s.t.

① $f_0 = \mathbb{1}_X$ ② $f_1(X) \subset A$ and ③ $f_t(A) \subset A$ $\forall t$.

ex Show that if X deformation retracts to A in this weak sense, then the inclusion $i : A \hookrightarrow X$ is a htpy equivalence

So you still get a homotopy equivalence even if you only have a weak deformation retraction.

eg. An example of a contractible space:
"house of 2 rooms" (see pg 4 Hatcher)

There are also some more interesting contractible spaces. To describe more spaces we need to talk about CW complexes.

Cell Complexes (aka. CW complexes)

(Lego for topologists)

(Intuitive definition here — fastest, by example)

0-cell $e^0 = *$

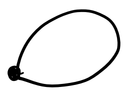
k-cell $\cong D^k \subset \mathbb{R}^k$

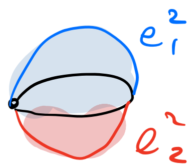
A cell complex is a space built by iteratively gluing cells to lower dimensional skeleta:

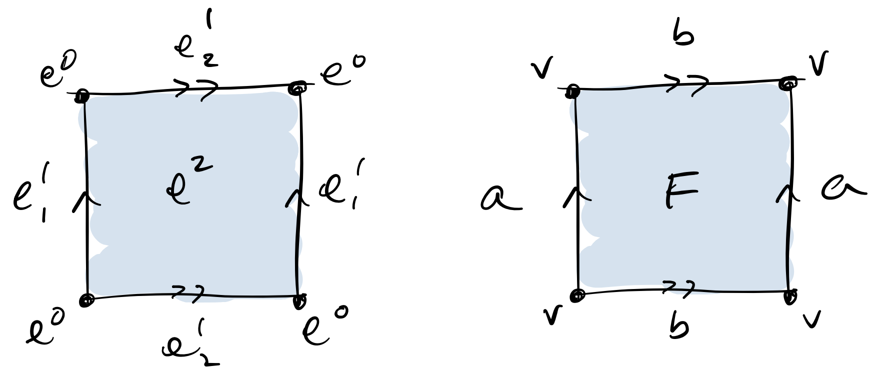
↪ quotient space

eg. I will build a space $X \cong S^2$

$X^0 = *$ just one 0 cell

$X^1 =$  $e^0 \cup e^1$ attach 1-cell to 0-skeleton X^0

$X^2 = S^2$ 

eg $T^2 = S^1 \times S^1 =$ 

better names ↑

- The data is all in the attaching maps:

$$\varphi_\alpha : S^{k-1} \cong \partial e^k_\alpha \longrightarrow X^{k-1}$$

- Can either

① Stop at a finite stage, so that $X = X^n$ for some $n < \infty$.

Quotient topology, from gluing.

Then n is called the <u>dimension</u> of X

and we say X is <u>finite-dimensional</u>.

— OR —

② Continue indefinitely, so that $X = \bigcup_n X^n$.

Then X is given the weak topology:

$A \subset X$ is open (resp closed) iff

$A \cap X^n$ is open (resp. closed) in X^n for each n.

<u>eg 0.1</u> A 1-dim'l cell cpx is called a graph: $X = X^1$

0-cells = "vertices", 1-cells = "edges"

<u>dfn</u>. The <u>Euler characteristic</u> $\chi(X)$ of a cell complex X is

{Important definition!}

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i \, \#(i\text{-cells})$$

ex) check χ of all CW decomps of S^1 in this lecture.

⊛ <u>Fact</u> / Thm 2.44 The χ of a cell cpx depends only on its homotopy type. Invariant of the <u>htpy type</u>

equiv class

eg. $\Rightarrow$ $S^1 \not\simeq *$ since $\chi(S^1) = 0$, $\chi(*) = 1$.

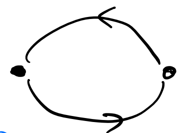
The spheres

$$S^n = \{ p \in \mathbb{R}^{n+1} \mid \|p\| = 1 \}$$

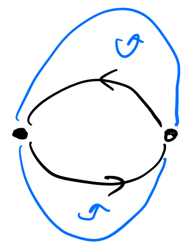
① Since $S^n \cong D^n / \partial D^n$, we see that we can build S^n using just two cells: $e^0 \cup e^n$ with the attaching map given by the constant map $S^{n-1} \to e^0$.

② There is another standard cell decomposition for spheres that is inductive:

$S^0 =$ • •

$S^1 =$ 

$= S^0 \cup 2$ 1-cells

$S^2 =$ 

$= S^1 \cup 2$ 2-cells.

...

using $\underbrace{\text{2 cells}}_{\text{hemispheres}}$ in each dimension up to n.

ex HW: describe these in coordinates carefully.
Key: attaching maps!

③ You can continue and build S^∞

$$S^\infty = \operatorname{colim}\left(S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow \cdots \right)$$

with 2 cells in each dimension $n \geq 0$.

Claim/Fact We will later use π_1 (fundamental group) to show that S^∞ is in fact contractible!