<u>Recall</u> A cell complex (CW cpx) is a space built by iteratively gluing cells to lower dim'l skeleta:

- start w/ collection of 0-cells $\rightsquigarrow$ 0-skeleton X^0
- attach all 1-cells to X^0 $\rightsquigarrow$ 1-skeleton X^1

...

- attach all n-cells to X^{n-1} $\rightsquigarrow$ X^n

Can either

① Stop at finite stage so that $X = X^n$, $n < \infty$.

Quotient topology from gluing.

Then n = the <u>dimension</u> of X,

& we say X is <u>finite dimensional</u>.

eg. {graphs} = {1-dim'l CW cpxs}

-OR-

② Continue indefinitely, so that $X = \bigcup_n X^n$

with the <u>weak topology</u>:

$A \subset X$ is open (resp. closed) iff

$A \cap X^n$ is open (resp closed) in X^n, $\forall n$.

eg. We can build successive S^n using two cells in each dimension, so that the k-skeleton of S^n is a copy of $S^k \subset S^n$.

$$S^\infty = \operatorname{colim}\left(S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow \cdots\right)$$

ex/HW Show that S^∞ is contractible!

eg. 0.4 (Quotient CW cpx example) Real Projective Space:

$\mathbb{R}P^n$ = moduli space of lines through $\vec{0}$ in $\mathbb{R}^{n+1}$

$= \mathbb{R}^{n+1} - \{0\} / x \sim \lambda x$ for $\lambda \in \mathbb{R} - \{0\}$.

$= S^n / x \sim -x$ $= S^n / -\mathbb{1}$ shorthand

- $-\mathbb{1} : S^n \longrightarrow S^n$, $x \mapsto -x$ is the antipodal map
- The antipodal map respects the standard cell decomposition on S^n: sends k-cells to k-cells. Quotient the CW cpx by $-\mathbb{1}$ to obtain a cell decomposition of $\mathbb{R}P^n$ with one cell in each dim $\leq n$.

ex/HW work out the details explicitly

eg. We can similarly define $\mathbb{C}P^n$, complex projective space = { complex lines through $\vec{0}$ in $\mathbb{C}^{n+1}$ }

Rmk There are spaces that are not htpy equiv to any CW cpx, eg. the long line.

Operations on Spaces

① (Cartesian) **Product**: If X, Y are cell cpxs, $X \times Y$ has cell structure induced by that of X and Y, with cells $e_\alpha^m \times e_\beta^n$

cell from X   cell from Y

⚠ Resulting topology may be finer than product topology, in extreme cases.

② **Quotient** by subcomplex $A \subset X$

X/A has cell structure of $X - A \cup *$

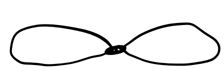
where all cells in A are crushed to $*$

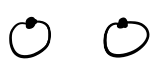
③ **Wedge** (aka. wedge sum)

Let $(X, x_0), (Y, y_0)$ be based spaces

space w/ a choice of basepoint

$$X \vee Y = X \sqcup Y / x_0 \sim y_0$$

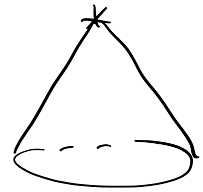
- eg. $S^1 \vee S^1$ = 



- can generalize to arbitrary collection of spaces:

$$\bigvee_\alpha X_\alpha$$

④ <u>Cone</u> of a space X:



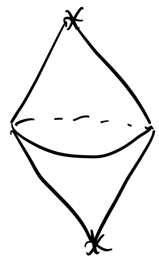
$$CX = (X \times I) / (X \times \{0\})$$

(ex) convince yourself that $CX \simeq *$.

⑤ (free) <u>suspension</u>

$$SX = (X \times I) / \underbrace{(x,0) \sim (y,0)}_{\text{all pts at } t=0 \text{ crushed to pt}}, \underbrace{(x,1) \sim (y,1)}_{\text{all pts at } t=1 \text{ crushed to pt}}$$

- e.g. SX where $X = S^1$



- In general $SX = CX \cup CX$ along the two copies of "$X \times \{0\}$"

(ex) Verify that $SS^n = S^{n+1}$

- If X is a CW cpx, SX has induced CW structure.

<u>Aside</u> SX is an important construction, esp later. "Stable htpy type" refers to stability under the suspension operation.

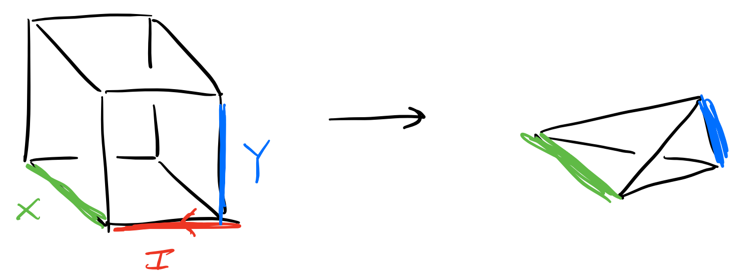
Maps can also be suspended: $Sf: SX \to SY$ is a quotient map of $f \times \mathbb{1} : X \times I \to Y \times I$.

⑥ Join

If we have two spaces X, Y we can define the space of all "line segments" joining pts in X to points in Y:

$$X * Y = X \times Y \times I \Big/ \begin{array}{l} (x, y_1, 0) \sim (x, y_2, 0) \\ \& \ (x_1, y, 1) \sim (x_2, y, 1) \end{array}$$

eg.



$X \times Y \times I$ $\longrightarrow$ $X * Y$

ex What is $X * pt$?

⑦ Smash product (more important later in alg. top)

$X \wedge Y = X \times Y / X \vee Y$ (X, Y based!)

a reduced version of $X \times Y$, where the canonical copies $X, Y \hookrightarrow X \times Y$ are crushed to a basepoint.

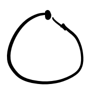
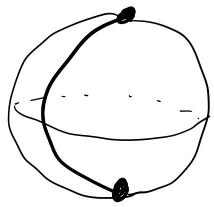
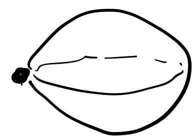
eg/ ex $S^1 \wedge S^1 = T^2 / S^1 \vee S^1 = S^2$

In fact, $S^m \wedge S^n = S^{m+n}$

Aside <u>Reduced suspension</u>: (X, x_0) based space

$$\Sigma X = SX \Big/ \underbrace{\{x_0\} \times I}_{\text{contractible}}$$

- If X is a cw cpx w/ x_0 a 0-cell, then $\Sigma X \simeq SX$ but has simpler cw structure:

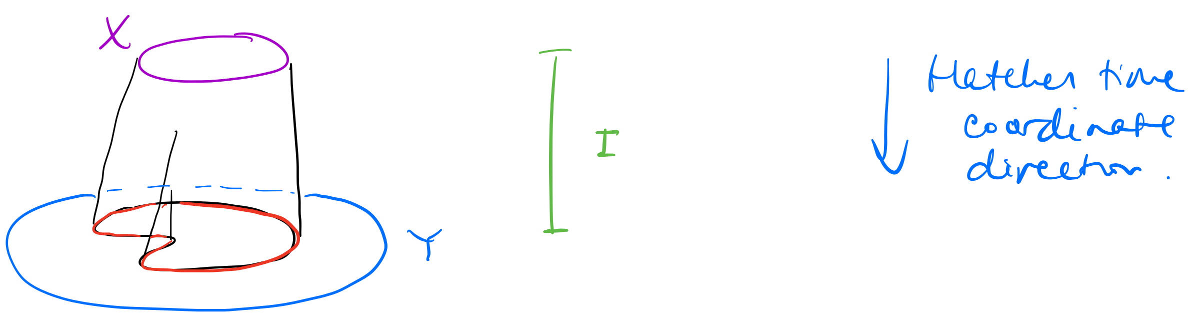
eg. $X =$  $SX =$  $\Sigma X =$ 

<u>eg./</u> ex $\quad X \wedge S^1 = \Sigma X.$

⊗ <u>mapping cylinder</u> (important!)

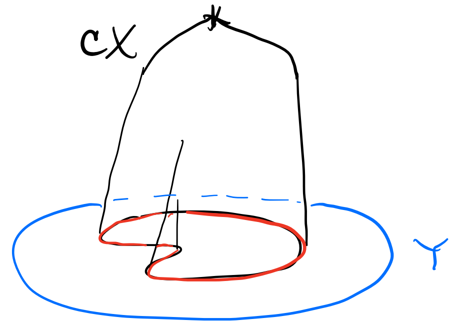
<u>defn.</u> For a map $f: X \to Y$, the mapping cylinder

$$M_f = (X \times I) \amalg Y \Big/ (x,1) \sim f(x)$$



ex Convince yourself that M_f deformation retracts to Y.

(9) Mapping cone $\quad f: X \to Y$

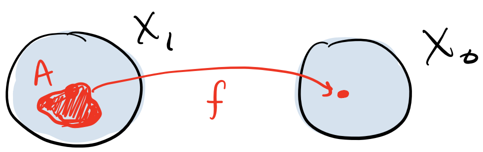


$$C_f = \underbrace{Y \amalg_f CX}_{\text{notation for attaching spaces:}} = M_f / X \times \{0\}$$

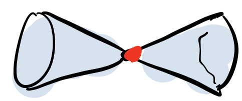
Notation Attaching spaces:

If we glue X_0, X_1 along a subspace $A \subset X_1$ via gluing map $f: A \to X_0$, we may write the result as $X_0 \amalg_f X_1$

e.g.



$\rightsquigarrow X_0 \amalg_f X_1 =$



Two Criteria for Homotopy Equivalence

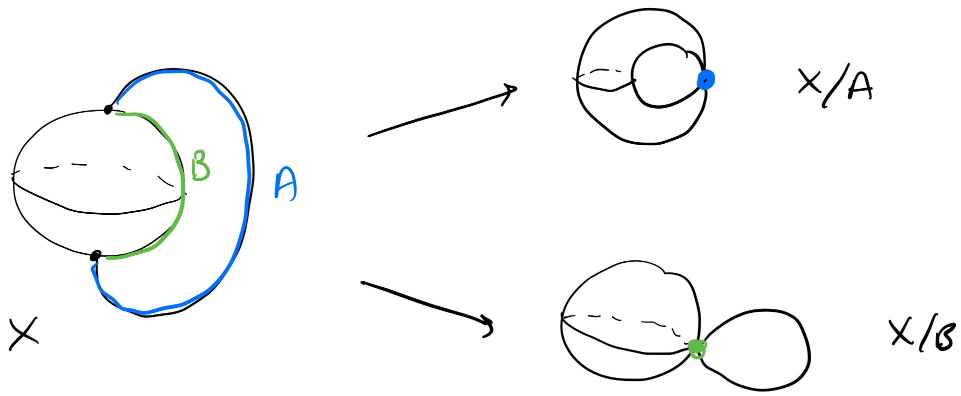
① Collapsing contractible subspaces:

Fact (proven later) If (X, A) is a CW pair w/ $A \simeq *$, then the quotient map $X \to X/A$ is a hpy equivalence.

ex Every graph with $|V|, |E| < \infty$ is $\simeq$ a disjoint union of wedges of circles

In Chp 1 we'll prove $\bigvee_n S^1 \not\simeq \bigvee_m S^1$ when $m \neq n$.

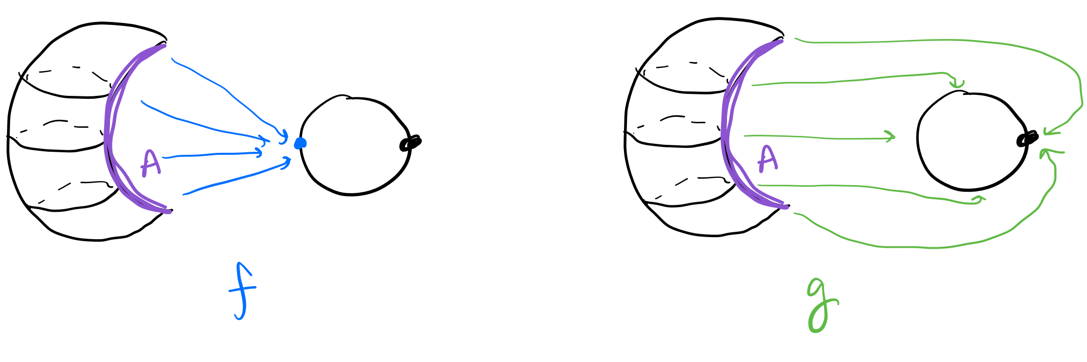
eg. Spaces can look very different!



② Varying attaching map

Fact (proven later) If (X, A) is a CW pair, and two attaching maps $f, g : A \to X_0$ are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$.

eg. (Compare with eg. above)



Homotopy Extension Property

Suppose we have a pair of spaces (X, A) — recall: "pair" means $A \subset X$

If for any ① $\tilde{f}_0 : X \to Y$

and ② htpy $f_t : A \to Y$ s.t. $\tilde{f}_0|_A = f_0$

we can find an "extension" $\tilde{f}_t : X \to Y$

i.e. where $\tilde{f}_t|_A = f_t$,

then we say that (X, A) satisfies / has the <u>homotopy extension property</u> (HEP).

In other words, (X, A) satisfies the HEP if

$\forall\, f_t, \tilde{f}_0$ in the diagram below, $\exists\, \tilde{f}_t$:

$$\begin{tikzcd}
 & & Y \\
A \times I \arrow[r] \arrow[urr, bend left, "\{f_t\}"] & X \times I \arrow[ur, dashed, "\exists \{\tilde{f}_t\}"] & \\
A \times \{0\} \arrow[r] \arrow[u, hook] & X \times \{0\} \arrow[u, hook] \arrow[uur, bend right, "\tilde{f}_0"'] &
\end{tikzcd}$$

We can describe HEP in terms of top spaces rather than space + time :

Claim A pair (X,A) has the HEP iff

$X\times\{0\}\cup A\times I$ is a retract of $X\times I$.

Pf.

$\boxed{\Rightarrow}$ HEP $\Rightarrow$

$$\begin{tikzcd}
X\times I \arrow[drr, "\exists\, \widetilde{id}"] & & \\
X\times\{0\}\cup A\times I \arrow[u, hook] \arrow[rr, "id"'] & & X\times\{0\}\cup A\times I
\end{tikzcd}$$

$$\begin{tikzcd}
& & Y \\
A\times I \arrow[r] \arrow[urr, bend left, "\{f_t\}"] & X\times I \arrow[ur, dashed, "\exists\,\{\tilde f_t\}"] & \\
A\times\{0\} \arrow[u, hook] \arrow[r] & X\times\{0\} \arrow[u, hook] \arrow[uur, bend right, "\tilde f_0"'] &
\end{tikzcd}$$

$\{f_t\} : A\times I \longrightarrow Y = X\cup\{0\}\cup A\times I$
is inclusion,

$\tilde f_0 : X\times\{0\} \longrightarrow Y = X\cup\{0\}\cup A\times I$
is also inclusion.

$\Rightarrow$ $\widetilde{id}$ is a map $X\times I \longrightarrow X\times\{0\}\cup A\times I$

that is id on $A\times I$, i.e. a deformation retract.

In particular, at time $t=1$, we have a retract.

$\boxed{\Leftarrow}$ Suppose $r: X\times I \longrightarrow X\times\{0\}\cup A\times I$ is a

retraction.

First suppose A is closed in X. (see remark at end)

Then any two maps

$$f \quad X\times\{0\} \longrightarrow X\times\{0\}\cup A\times I$$

$$g: \quad A\times I \longrightarrow X\times\{0\}\cup A\times I$$

that agree on $A\times\{0\} = (X\times\{0\})\cap(A\times I)$

glue to form a map

$$F : X\times\{0\} \cup A\times I \longrightarrow X\times\{0\} \cup A\times I$$

i.e. defined by $F|_{X\times\{0\}} = f$, $F|_{A\times I} = g$.

$$\begin{tikzcd}
 & X\times\{0\} \arrow[dr] \arrow[drr, bend left, "f"] & & \\
A\times\{0\} \arrow[ur] \arrow[dr] & & X\times\{0\} \cup A\times I \arrow[r, dashed, "F"] & Y \\
 & A\times I \arrow[ur] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

Note that F is continuous since it's cont. on the closed sets $X\times\{0\}$, $A\times I$.

Then $F\circ r$ is an extension $X\times I \to Y$.

(of the map f & htpy g, the input data for the HEP.)

//

Rmk. What if A is not closed?

- See argument in Appendix that generalizes the proof for arbitrary A.
- Observe that if X is Hausdorff, ($\to$ eg X is a CW cpx) then $A\subset X$ is necessarily closed, in this context:

$$X\times\{0\}\cup A\cup I = \operatorname{img} r = \{z\in X\times I \mid r(z)=z\}$$

is closed since X Hausdorff

$$\Rightarrow A\times\{1\} = (\underbrace{X\times\{0\}\cup A\cup I}_{\text{closed}}) \cap (X\times\{1\})$$

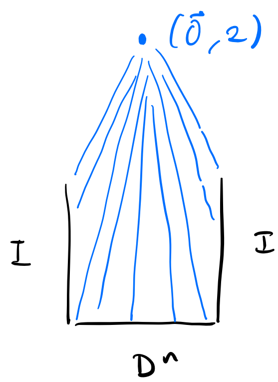
is closed in $X\times\{1\}$. //

Prop. 0.16 If (X,A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$.

(Hence (X,A) has the HEP.)

Pf. Sketch

- First note $D^n \times I$ def retracts to $D^n \times \{0\} \cup \partial D^n \times I$ by $r_t = tr + (1-t)\mathbb{1}$ where r is projection from $(\vec{0}, 2)$:



- At the n-skeleton, we can deformation retract

$$X^n \times I \to X^n \times \{0\} \cup A^n \times I$$

by simultaneously applying def. retractions like above for each n-cell not in A.

- Combine all these htpies by performing the def retraction for the n skeleton during $[1/2^{n+1}, 1/2^n]$.

- Continuity at 0? Any point in $X \times I$ is in some $X^n \times I$.

$\Rightarrow$ it is stationary on $[0, 1/2^{n+1}]$.

(Recall CW cpxs have the weak topology with respect to the skeleta)

Prop 0.17 If (X, A) has the HEP and $A \simeq *$, then the quotient map $q: X \to X/A$ is a htpy equiv.

Pf.

- A is contractible $\Rightarrow$ $\exists$ htpy $f_t : A \to X$ where f_0 = inclusion of A, f_1 = const map onto some point $a \in A \subset X$

  (start w/ $f_t : A \to A$, extend target)

- (X, A) has HEP $\Rightarrow$ $\exists$ extension $\tilde{f}_t : X \to X$ where $\tilde{f}_0 = id_X$ and $\tilde{f}_t|_A = f_t$.

- Since $q\tilde{f}_t : X \to X/A$ sends $A \to pt$, it factors as $\bar{f}_t \cdot q$: (see notes on comm diag, "factors through")

$$\begin{tikzcd} X \arrow[r, "\tilde{f}_t"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{f}_t"'] & X/A \end{tikzcd}$$

$\bar{f}_t$ ← well-defined

At $t=1$, $\tilde{f}_1(A) = a$, so $\tilde{f}_1$ factors as $\tilde{f}_1 = gq$

(g is well defined)

$$\begin{tikzcd} X \arrow[r, "\tilde{f}_1"] \arrow[d, "q"'] & X \\ X/A \arrow[ur, "g"'] & \end{tikzcd}$$

We have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\
X/A \arrow[r, "\bar{f}_1"'] \arrow[ur, "g"] & X/A
\end{tikzcd}$$

top triangle commutes by definition of g

Bottom triangle:

any x where $q(x) = \bar{x}$: any "lift"

$$qg(\bar{x}) = qgq(x) = q\tilde{f}_1(x) = \bar{f}_1 q(x) = \bar{f}_1(\bar{x}).$$

- Now check that g and q are inverse homotopy equivalences:
  - $gq = \tilde{f}_1 \simeq \tilde{f}_0 = \mathbb{1}_X$ via $\tilde{f}_t$
  - $qg = \bar{f}_1 \simeq \bar{f}_0 = \mathbb{1}_{X/A}$ via $\bar{f}_t$