(Chp 0 cont'd) OH M, F MSB 3240

<u>defn.</u> A htpy $f_t : X \to Y$ whose restriction to $A \subset X$ is independent of t (i.e. $f_t|_A = f_0|_A \ \forall t$) is called a <u>htpy relative to A</u> (<u>htpy rel A</u>).

eg. A deformation retraction $f_t : X \to X$ onto $A \subset X$ is a htpy rel A, since $f_t|_A$ = inclusion of A. ("$\mathbb{1}$").

<u>Prop 0.19</u> Suppose (X,A) and (Y,A) satisfy the HEP and $f : X \to Y$ is a htpy equiv w/ $f|_A = \mathbb{1}$ (interpreted appropriately). Then f is a $\underbrace{\text{htpy equiv rel } A}_{}$.

($\exists$ htpies rel A)

(proof in discussion)

<u>Cor 0.20</u> If (X,A) has HEP and $A \hookrightarrow X$ is a htpy equiv, then A is a deformation retract of X

<u>Cor 0.21</u> A map $f : X \to Y$ is a htpy equiv <u>iff</u> X is a deformation retract of the mapping cylinder M_f.

$\Longrightarrow$ <u>Cor</u> X, Y are htpy equiv <u>iff</u> $\exists$ 3rd space containing both X and Y as def. retracts.

(recall: already know $M_f \xrightarrow{\text{def retr}} Y$.)

Pf sketch $\boxed{\Leftarrow}$ clear. $\boxed{\Rightarrow}$:

$$\begin{tikzcd} & Y \arrow[d, shift right, "j"'] \\ X \arrow[ur, "f"] \arrow[r, "i"'] & M_f \arrow[u, shift right, "r"'] \end{tikzcd}$$

- All canonical maps; i, j are inclusion; r is canonical retraction.

- check that the diagram to the left "homotopy commutes" i.e. commutes up to homotopy.
  - $f = ri$
  - $i \simeq jf$ by slide.

- since j and r are htpy equivalences, f is a htpy equiv iff i is (& it is by hypothesis)

Claim (M_f, X) satisfies the HEP
- see Example 0.15 in Discussion
- or consider case of CW cpxs.

So by Cor 0.20, X is a def retract of M_f. ∎

Toward Fundamental Group

We want to define an assignment

$$\begin{tikzcd}[column sep=huge] \text{topology} \\ \text{top spaces} \arrow[r] \arrow[d] & \text{Groups} \\ \frac{\text{top spaces}}{\text{homotopy equiv}} \arrow[ur] & \end{tikzcd}$$

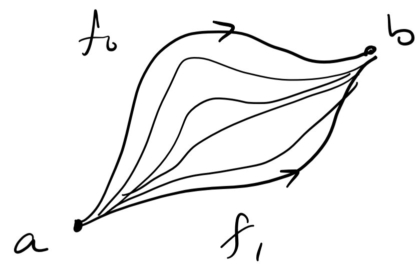
algebra

X $\quad \pi_1(X)$

So that we can use algebra to study top spaces.

#

Paths + Homotopy



- path in X from a to b:

  $f : I \to X$ where $f(0) = a$
  $\qquad\qquad\qquad\qquad f(1) = b$.

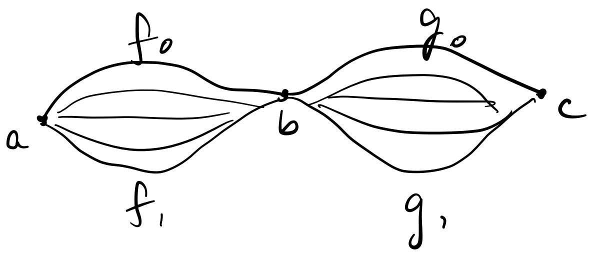
- htpy of paths : $\{f_t : I \to X\}$ where $f_t(0) = a$
  $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f_t(1) = b$.

  Associated $F : I \times I \to X$ is continuous.

- We write $[f]$ for the equivalence class of f under homotopy. $\quad f_0 \simeq f_1$ above.

⚠ notation! • We have composition $f \bullet g$ (aka product path) that traverses f first then g, by reparametrizing

$$f \bullet g(s) = \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$



ex Convince yourself that we have well defined composition on htpy classes:

$$\underbrace{[f]\bullet[g]}_{\text{Hatcher writes } [f][g]} = [f \bullet g].$$

Fundamental group

$x_0 \in X$. A loop based at x_0 is a path $f: I \to X$ where $f(0) = f(1) = x_0$.

The fundamental group of the based space (X, x_0) is
- the set of all htpy classes of loops in X based at x_0
- with product (group operation) $[f][g] = [f \bullet g]$.

and is denoted $\pi_1(X, x_0)$.

prop. $\pi_1(X, x_0)$ is a group.

Pf. check identity, (2-sided) inverses ex

$c(t) = x_0$ $\quad f \mapsto \bar{f}$

(ex) $X \subset \mathbb{R}^n$ convex then $\pi_1(X, x_0) = 1$

π_1 is in general not abelian so the multiplicative identity should be written as "1".

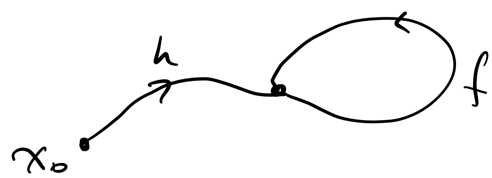
<u>Rmk</u>. $\pi_n(X, x_0)$ is defined similarly, but by considering maps $f : I^n \longrightarrow X$ where $f(\partial I^n) = x_0$. These are the <u>higher homotopy groups</u> which we may discuss later.

<u>Observe</u> we can move the basepoint:

Let h be a path from x_0 to x_1.

Then we can define the <u>change of basepoint homomorphism</u>

$$\beta_h : (X, x_0) \longrightarrow (X, x_1)$$
$$[f] \longmapsto [h \cdot f \cdot \bar{h}]$$



(ex) • Check β_h is well defined

• check β_h is indeed a homomorphism.

<u>Prop</u> β_h is an isomorphism

pf. (ex) inverse is $\beta_{\bar{h}}$ (check)

Rmk. If X is path-cntd, then the isomorphism class of $\pi_1(X, x_0)$ doesn't depend on the choice of basepoint x_0; then we can write $\pi_1(X)$ or $\pi_1 X$.

defn A space X is simply-connected if it is path cntd and has $\pi_1(X) = 1$.

eg. $X = *$, S^n where $n \geq 2$.

noneg: $X = Y \cup T \subset \mathbb{R}^2 \cup \infty = S^2$ where Y = y-axis $\cup \infty$,
T is the topologist's sine curve $\{y = \sin(\frac{1}{x}) \mid x \in (0, 2/\pi)\}$
then X is connected but not path cntd.
If $x_0 \in Y$ then $\pi_1(X, x_0) \cong \mathbb{Z}$ (as we'll see)
but if $x_0 \in T$ then $\pi_1(X, x_0) = 1$.

prop 1.6 A space X is simply-cntd iff
$\forall\, x, y \in X$, $\exists!$ htpy class of paths from x to y.

pf.
$\boxed{\Rightarrow}$ If f, g are paths $x \rightsquigarrow y$, then $f \simeq f\bar{g}g \simeq g$
since $[f\bar{g}] = 1 \in \pi_1(X)$, ie $f\bar{g} \simeq c$ (const).
$\boxed{\Leftarrow}$ $\exists!$ class of paths from x_0 to x_0. ∎

Toward Fundamental Group of the Circle.

thm 1.7 $\pi_1(S^1) \cong \mathbb{Z}$, generated by the htpy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ based at $(1,0)$.

Let $* = (1,0)$

$\omega(s) = (\cos 2\pi s, \sin 2\pi s)$

$\omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s) \qquad n \in \mathbb{Z}$.

Note that $[\omega]^n = [\omega_n]$.

$\mathbb{R} \to \mathbb{R}^3$

$s \mapsto (\cos 2\pi s, \sin 2\pi s, s)$

projection $\quad p \downarrow$

(x,y,z)

$\downarrow$

(x,y)

(parking garage)

$\tilde{\omega}_n : I \to \mathbb{R}$ is the path winding around the
$s \mapsto ns$

helix $|n|$ times (up if $n>0$, down if $n<0$)

note: $*$ $\tilde{\omega}_n$ starts at $0 \in \mathbb{R}$!

note $\omega_n = p\tilde{\omega}_n$

$$\begin{tikzcd} & \mathbb{R} \arrow[r, hook] \arrow[d, "p"] & \mathbb{R}^3 \\ I \arrow[ur, "\tilde{\omega}_n"] \arrow[r, "\omega_n"'] & S^1 & \end{tikzcd}$$

we say $\tilde{\omega}_n$ is a lift of ω_n.

Covering Spaces Intro

<u>defn.</u> Let X be a space.

A <u>covering space of X</u> is a space $\tilde{X}$ together with a map $p : \tilde{X} \longrightarrow X$ satisfying the "covering space condition":

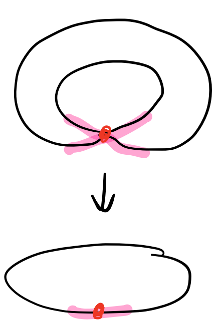
stack of pancakes

For each pt $x \in X$, $\exists$ open nbhd U of x in X s.t. $p^{-1}(U)$ is a union of <u>disjoint</u> open sets, each of which is mapped homeomorphically onto U by p "<u>evenly covered</u> nbhd U"

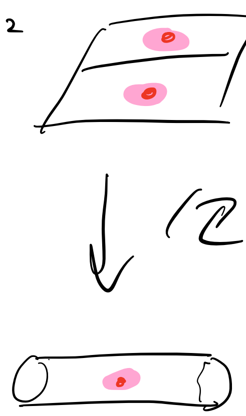
eg. $\mathbb{R}$

• = the $\mathbb{Z}$-many lifts $\tilde{x}$ of x

<u>non eg</u>

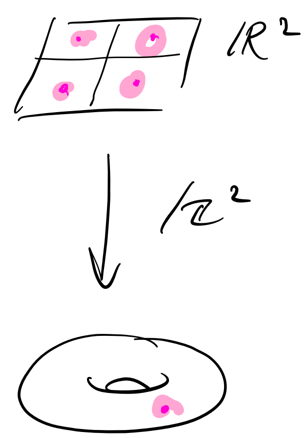


eg. $\mathbb{R}^2$



∞ cyl

eg.



Fact We will use to prove $\pi_1 S^1 \cong \mathbb{Z}$

(a) For each path $f: I \to X$ starting at $x_0 \in X$,

and each $\tilde{x}_0 \in p^{-1}(x_0)$,

there is a unique lift $\tilde{f}: I \to \tilde{X}$ starting at $\tilde{x}_0$.

(b) For each hpty $f_t: I \to X$ of paths starting at x_0,

and each $\tilde{x}_0 \in p^{-1}(x_0)$

there is a unique lifted hpty $\tilde{f}_t: I \to \tilde{X}$

of paths starting at $\tilde{x}_0$.

Leave proofs for later. But in our case maybe can believe!

pf of thm 1.7 assuming facts (a) & (b)

Let $f: I \to S^1$ be a loop based at $x_0 = (1, 0)$ WTS $[f] = [\omega_n]$ for some n.

- By (a), $\exists!$ lift $\tilde{f}$ starting at $0 \in \mathbb{R}$
  - This path ends at some $n \in \mathbb{Z}$ since $p(\tilde{f}(1)) = x_0$.

    and $p^{-1}(x_0) = \mathbb{Z}$.
- Other path from $0 \rightsquigarrow n$ in $\mathbb{R}$ is $\tilde{\omega}_n$.
  - $\tilde{f} \simeq \tilde{\omega}_n$ via the linear homotopy $(1-t)\tilde{f} + t\tilde{\omega}_n$
  - Then $(1-t)p\tilde{f}_n + tp\tilde{\omega}_n$ is a hpty $f \simeq \omega_n$

    $\Rightarrow [f] = [\omega_n]$.

Now WTS n is uniquely determined by $[f]$.

Suppose $\omega_m \simeq f \simeq \omega_n$. Let g_t be a htpy from ω_m to ω_n.

- By (b) g_t lifts to htpy $\tilde{g}_t$ of paths starting at $0 \in \mathbb{R}$.
- (a) $\Rightarrow$ $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$ (uniqueness)
- Since $\tilde{f}_t$ is a htpy of paths, the endpoints of $\tilde{\omega}_m$ and $\tilde{\omega}_n$ must be the same. ∎

FRIDAY — put (a), (b) on board before class.

To prove (a) & (b), we prove a more general statement:

Claim (c) Given $F : Y \times I \to X$

and $\tilde{F}_0 : Y \times \{0\} \to \tilde{X}$ lifting $F|_{Y \times \{0\}}$,

there exists a unique map $\tilde{F} : Y \times I \to X$ lifting F

and restricting to the given $\tilde{F}_0$ on $Y \times \{0\}$.

That's a mouthful! Clearer in diagram form:

Hatcher pf: abuse notation & write F for restrictions of $\tilde{F}$

$$\begin{tikzcd}
 & & \tilde{X} \arrow[d, "p"] \\
Y \times \{0\} \arrow[r, hook, "\text{inclusion}"'] \arrow[urr, "\tilde{F}_0"] & Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists!\, \tilde{F}"] & X
\end{tikzcd}$$

covering map

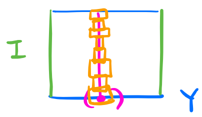
(a) For each path $f : I \to X$ starting at $x_0 \in X$, and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f} : I \to \tilde{X}$ starting at $\tilde{x}_0$.

(a)
$$\begin{tikzcd}
 & & \tilde{X} \arrow[d, "p"] \\
\{0\} \arrow[r, hook] \arrow[urr, "\tilde{x}_0"] & I \arrow[r, "f"'] \arrow[ur, "\exists!\, \tilde{f}"] & X
\end{tikzcd}$$

(b) For each htpy $f_t : I \to X$ of paths starting at x_0, and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted htpy $\tilde{f}_t : I \to \tilde{X}$ of paths starting at $\tilde{x}_0$.

(b)
$$\begin{tikzcd}
 & & \tilde{X} \arrow[d, "p"] \\
I \times \{0\} \arrow[r, hook] \arrow[urr, "\tilde{f}_0"] & I \times I \arrow[r, "f_t"'] \arrow[ur, "\exists!\, \tilde{f}_t"] & X
\end{tikzcd}$$

pf of ⓒ



* very point-set proof. we will go through quickly.

(1) 1st construct a lift $\tilde{F} : N \times I \to \tilde{X}$ for $y \in N \subset Y$ nbhd.

Since $F : Y \times I \to X$ is cts,

every pt $(y_0, t) \in Y \times I$ has a product nbhd

$(y_0, t) \subset N_t \times (a_t, b_t)$ (N_t for now)

small enough s.t. $F(N_t \times (a_t, b_t))$ is contained

in an evenly covered nbhd of $F(y_0, t)$.

- Since $\{y_0\} \times I$ is compact, finitely many such $N_t \times (a_t, b_t)$ cover the interval $\{y_0\} \times I$.

$\Rightarrow$ we can choose

- a single nbhd $y_0 \in N \subset Y$ to act as N_t for each nbhd $N_t \times (a_t, b_t)$ in this finite set
- $0 = t_0 < t_1 < \dots < t_m = 1$ s.t. $\forall i$,

  $F(N \times [t_i, t_{i+1}]) \subset U_i$ evenly covered.

- Assume inductively that $\tilde{F}$ has already been constructed on $N \times [0, t_i]$, agreeing with the given $\tilde{F}$ on $N \times \{0\}$.

- $F(N \times [t_i, t_{i+1}]) \subset U_i$ even covered

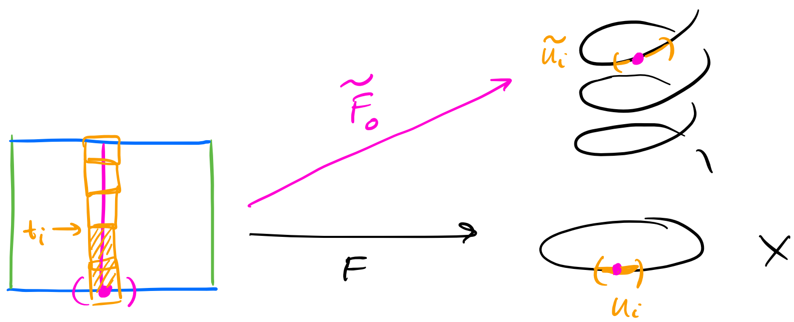
  $\Rightarrow \exists !$ lift $\tilde{U}_i \subset \tilde{X}$ proj homeomorphically onto U_i

  nbhd of $\tilde{F}(y_0, t_i)$

- After shrinking N possibly we may assume $\tilde{F}(N \times \{t_i\})$ is contained in $\tilde{U}_i$

  i.e. replace $N \times \{t_i\}$ by

  $N \times \{t_i\} \cap (\tilde{F}|_{N \times \{t_i\}})^{-1}(\tilde{U}_i)$



- Now define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ to be the composition of F with the homeo $p^{-1} : U_i \to \tilde{U}_i$.

  (lift F on this nbhd)

- When i reaches m we are done constructing $\tilde{F}$ on the nbhd N of y_0!

② Uniqueness. (sketch)

Prove case where $Y = \{y_0\}$

Idea: If $\tilde{F}, \tilde{F}'$ are two lifts,

since (inductively) $\tilde{F}(t_i) = \tilde{F}'(t_i)$,

$\tilde{F}([t_i, t_{i+1}]), \tilde{F}'([t_i, t_{i+1}]) \subset$ the same $\tilde{U}_i$.

Since p is injective on $\tilde{U}_i$ and $p\tilde{F} = p\tilde{F}'$,

we have $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$.

When Y is more than a point, note that $\tilde{F}$ is unique on each $N \times I$ when restricted to each line segment $\{y\} \times I$. $\Rightarrow$ they must agree when two $N \times I$'s overlap.

$\Rightarrow$ get a well defined lift $\tilde{F}$ on all of $Y \times I$, continuous b/c cont on each $N \times I$, unique b/c unique on each $\{y\} \times I$. $\blacksquare$

Applications of the fact $\pi_1(S^1) \cong \mathbb{Z}$.

thm 1.8 (Fundamental Theorem of Algebra)

Every nonconstant polynomial in $\mathbb{C}[z]$ has a root in $\mathbb{C}$.

Pf sketch

- Suppose that $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ has no roots in $\mathbb{C}$. (WTS $p(z)$ is constant.)
- For each $r \in \mathbb{R}$,

$$f_r(s) = \frac{p(re^{2\pi i s}) / p(r)}{|p(re^{2\pi i s}) / p(r)|}$$

is a loop in S^1 based at $1 \in \mathbb{C}$.

(htpy of loops)

- notice $f_0(s)$ is the trivial loop.

(Maybe in Discussion)

Borsuk-Ulam Theorem in dim 2

"$\forall t$, $\exists$ antipodal pts on Earth w/ same temp + pressure"

<u>thm. 1.10</u> For every continuous map $f: S^2 \to \mathbb{R}^2$, there exists a pair of antipodal points $\{x, -x\}$ in S^2 with $f(x) = f(-x)$.

eg. maybe only one: eg. projection S^2 to $\mathbb{R}^2$.

ex Try proving to yourself why this holds in dim 1: $f: S^1 \to \mathbb{R}$. $g(x) = f(x) - f(-x)$

Pf.

- Suppose BWOC that $\exists$ $f: S^2 \to \mathbb{R}^2$ where $\forall x$, $f(x) \neq f(-x)$. Then we may define a fn

$$g(x) = \frac{f(x) - f(-x)}{\| f(x) - f(-x) \|} : S^2 \to S^1 \subset \mathbb{R}^2$$

$*$ Observe that $g(x) = -g(-x)$

- Define a loop $\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0) : I \to \mathbb{R}^3$

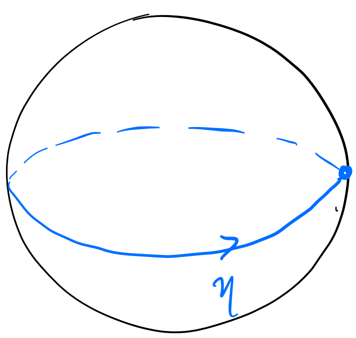
this is ω from last class, but into $\mathbb{R}^3$

Let $h(s)$ be the composed loop $g \circ \eta : I \to S^1$

interpreted suitably:

$$I \xrightarrow{\eta} S^1 \hookrightarrow S^2 \xrightarrow{g} S^1$$

loop? ✓.



- Since $g(-x) = -g(x)$, we have
$$h(s+1/2) = -h(s) \quad \text{for } s \in [0, 1/2]$$
Lift the loop h to $\tilde{h} : I \to \mathbb{R}$ (based at 0, say)
$\Rightarrow \tilde{h}(s+1/2) = \tilde{h}(s) + q/2 \qquad q \in 2\mathbb{Z}+1$

q depends continuously on $s \Rightarrow$ independent of s

$\Rightarrow h(1) = \tilde{h}(1/2) + q/2 = (\tilde{h}(0) + q/2) + q/2 = \tilde{h}(0) + q.$

$\Rightarrow [h] = [\omega_q] \neq [0]$ in $\pi_1(S^1)$ (as q is odd, $\neq 0$)

- OTOH, $h = g \circ \eta$ when η was obviously null homotopic in $S^2 \Rightarrow g \circ \eta$ is nullhomotopic (by comp of htpy) ↯.

"Contradiction"

∎