π_1 cont'd; some computational techniques 10.13.25

- Relationships b/w spaces, induced homomorphisms

prop 1.12 If X and Y are path connected, then

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

generally

pf.

Product topology on $X \times Y$:

$f : Z \to X \times Y$ is continuous

iff $g : Z \to X$, $h : Z \to Y$

defined by $f(z) = (g(z), h(z))$

are both continuous

$$\begin{tikzcd} & & X \\ Z \arrow[r, "f"] \arrow[urr, bend left, "g"] \arrow[drr, bend right, "h"'] & X \times Y \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & & Y \end{tikzcd}$$

all projections are continuous

$\Rightarrow$ loop $f : I \to X \times Y$ based at (x_0, y_0) is equiv to a pair of loops g in X based at x_0, h in Y based at y_0

Same with a htpy $\{f_t\} : I \times I \to X \times Y$.

$\Rightarrow$ get a bijection

$$\pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$$
$$[f] \longmapsto ([g], [h])$$

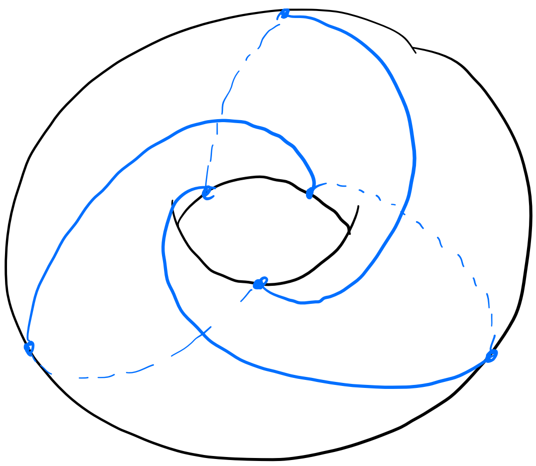
ex easy to check this is a homomorphism $\Rightarrow$ isom. $\blacksquare$

can do generally

e.g. T^n = n-dim'l torus = $\underbrace{S^1 \times \dots \times S^1}_{n}$

$\pi_1(T^n) \cong \mathbb{Z}^n = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{n}$

e.g. In T^2 here is a loop representing $(3,2) \in \pi_1(T^2)$:



torus knots

Induced Homomorphisms

Let $\varphi : (X, x_0) \longrightarrow (Y, y_0)$ be a map of based spaces
ie $\varphi(x_0) = y_0$

Then φ induces a homomorphism

$$\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$
$$[f] \longmapsto [\varphi \circ f]$$

Check:

- well-defined?

  If $f_0 \simeq f_1$ via f_t, then $\varphi \circ f_t$ is a htpy relating $\varphi \circ f_0 \simeq \varphi \circ f_1$.

- homomorphism?

$$\varphi(f \cdot g) = (\varphi \circ f) \cdot (\varphi \circ g)$$
$$= \begin{cases} \varphi \circ f(s) & s \in [0, 1/2] \\ \varphi \circ g(s) & s \in [1/2, 1] \end{cases}$$

Functoriality of π_1 Henceforth $\varphi \circ f$ written φf (clean)

① $(\varphi \psi)_* = \varphi_* \psi_*$ b/c $(\varphi \psi) f = \varphi(\psi f)$

$$(X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$$

② $\mathbb{1}_* = \mathbb{1}$ (identity map on spaces $\rightsquigarrow$ identity hom)

$\Rightarrow$ $\pi_1(-) : \mathrm{Top}_* \longrightarrow \mathrm{Grp}$ is a functor.

Cor: If $X \underset{\psi}{\overset{\varphi}{\rightleftarrows}} Y$ are inverse homeomorphisms, then φ_*, ψ_* are inverses.

$\varphi_* \psi_* = (\varphi\psi)_* = \mathbb{1}_*$, same for $\psi_* \varphi_*$

We'll use this to prove $\pi_1(S^n) = 0$ if $n \geq 2$.

Lemma 1.15 If $X = \bigcup A_\alpha$ where each A_α

- is open, path cntd and contains the basepoint x_0,
- & each $A_\alpha \cap A_\beta$ is path cntd,

then every loop in X based at x_0 is homotopic to a product of loops, each of which is contained in some A_α.

pf. (idea)

- Let $f: I \to X$ be a loop.
- $\exists$ partition $0 = s_0 < s_1 < \dots < s_m = 1$ s.t. $[s_i, s_{i+1}] \subset A_{\alpha_i}$
    - take collection of nbhds each contained in an A_α
    - I compact $\Rightarrow$ only need finitely many intervals to cover.
- Write f_i for the piece of the path $f|_{[s_i, s_{i+1}]}$, reparametrized to $f_i : I \to A_{\alpha_i} \subset X$
- For each i, pick path g_i in $A_{\alpha_i} \cap A_{\alpha_{i+1}}$

from $x_0 \rightsquigarrow f(s_i)$.

- Then the loop

$$(f_1 \cdot \bar{g}_1) \cdot (g_1 f_2 \bar{g}_2) \cdot \dots \cdot (g_{m-1} \cdot f_m) \simeq f.$$

■

<u>prop 1.14</u> $\pi_1(S^n) = 0 \quad n \geq 2.$

pf.

- we can write $S^n = A \cup B$ where

$A = S^n \setminus \{\text{south pole}\}$

$B = S^n \setminus \{\text{north pole}\}.$

Let $x_0 = (1, 0, \dots, 0)$ $\neq$ north pole, south pole

- Loop f based at x_0 is hpc to one in the form of Lemma 1.15.
- But $\pi_1(A) \cong \pi_1(B) \cong 1$.

so we can (simultaneously) homotope each subloop to the constant path. ■

<u>Cor.</u> $\mathbb{R}^2$ is <u>not</u> homeomorphic to $\mathbb{R}^n$ for $n \neq 2$

pf.

- Clearly $\mathbb{R}^2 \not\cong \mathbb{R}^0$ path cntdness
- $\not\cong \mathbb{R}^1$ $\mathbb{R}^2 - \{0\}$ path cntd, $\mathbb{R}^1 - \{f(0)\}$ not path cntd.

If $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^n$, consider induced homeo

$$\mathring{f}: \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{f(0)\}.$$

Then $\mathring{f}_*: \pi_1(\mathbb{R}^2 \setminus \{0\}) \longrightarrow \pi_1(\mathbb{R}^n \setminus \{f(0)\})$

cannot be an isom:

- $\mathbb{R}^2 \setminus \{0\} \simeq S^1 \times \mathbb{R}$

  $\Rightarrow \pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$

- $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1} \times \mathbb{R}$

  $\Rightarrow \pi_1(\mathbb{R}^n \setminus \{0\}) \cong \pi_1(S^{n-1}) \times \pi_1(\mathbb{R}) = 1.$

$\boxtimes$

WEDNESDAY

Prop 1.17

- If X retracts to $A \subset X$, then $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$ is injective.
- If X deformation retracts to A, then i_* is an isom.

Pf.

- $r: X \to A$ retraction $\Rightarrow ri = \mathbb{1} \Rightarrow r_* i_* = \mathbb{1}$

  $\Rightarrow i_*$ injective.

- $r_t: X \to X$ def. retraction. $(r_1(X) \subset A)$

  Any loop $f: I \to X$ based at $x_0 \in A$ is htpic to a loop $r_1 f$ in A via $r_t f$

  $\Rightarrow i_*$ is also surjective.

$\boxtimes$

Cor $S^1 = \partial D^2$ is not a retract of D^2.

Homotopy Equivalences and π_1 (concerns about basepoint?)

<u>defn.</u> <u>Homotopy of pairs</u>: $\varphi_t : (X, A) \to (Y, B)$ means $\forall t$, $\varphi_t(A) \subset B$.

In particular, if $A = \{x_0\}$, $B = \{y_0\}$, we have a <u>basepoint-preserving homotopy</u>

$\rightsquigarrow$ use this to define <u>htpy equiv of based spaces</u>.

To handle homotopies that do <u>not</u> fix the basepoint: image of x_0 moves with time t:

- <u>Lemma 1.19</u> Let $\varphi_t : X \to Y$ be a htpy, and $h(t) = \varphi_t(x_0)$ a path in Y. Then the following diagram commutes:

$$\begin{tikzcd} & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\beta_h"] \\ \pi_1(X, x_0) \arrow[ur, "\varphi_{1*}"] \arrow[r, "\varphi_{0*}"'] & \pi_1(Y, \varphi_0(x_0)) \end{tikzcd}$$

Pf. • Let h_t be the (reparametrized) restriction of h to the interval $[0,t]$: $h_t(s) = h(ts)$

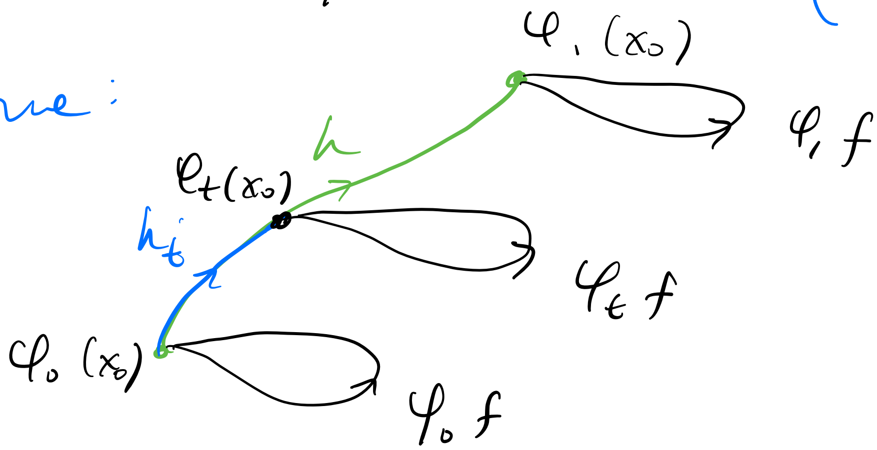
• If f is a loop in X at basepoint x_0, then $h_t \cdot (\varphi_t f) \cdot \overline{h_t}$ gives a htpy of loops based at $\varphi_0(x_0)$.

• Observe that then

$$\varphi_{0*}([f]) = \beta_h(\varphi_{1*}([f])).$$

(The equation given by the diagram)

Picture:



∎

prop 1.18 If $\varphi: X \to Y$ is a htpy equivalence then $\varphi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \varphi(x_0))$ is an isomorphism $\forall$ choices $x_0 \in X$.

pf

Let $\psi: Y \to X$ be a htpy inverse for φ.

Recall: $\varphi\psi \simeq \mathbb{1}$, $\psi\varphi \simeq \mathbb{1}$.

Consider:

$$\underbrace{\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\varphi(x_0))}_{\psi\varphi \simeq \mathbb{1}_X} \xrightarrow{\varphi_*} \pi_1(Y, \varphi\psi\varphi(x_0))$$

$\Rightarrow \exists$ htpy $f_t: X \longrightarrow X$ from $\psi\varphi$ to $\mathbb{1}_X$.

$\Rightarrow$ By Lemma, $f_{0*} = \beta_h f_{1*}$ for some path h

$\psi_*\varphi_*$ $\mathbb{1}_*$

$\Rightarrow \psi_*\varphi_* = \beta_h$

isomorphism (as we showed before)

$\Rightarrow \psi_*\varphi_*$ is isom $\Rightarrow \varphi_*$ is injective.

Similarly,

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \underbrace{\pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi\psi\varphi(x_0))}_{\varphi\psi \simeq \mathbb{1}_Y}$$

$\Rightarrow \varphi_*\psi_* = \beta_{h'}$ h' a path in Y

isom

$\Rightarrow \varphi_*$ is surjective.

//

* Technically φ_* here is based at $\psi\varphi(x_0)$.

To distinguish from the previous $\varphi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \varphi(x_0))$, let's denote the second instance as

$$\varphi_*': \pi_1(X, \psi\varphi(x_0)) \longrightarrow \pi_1(Y, \varphi\psi\varphi(x_0)).$$

Since $\psi\varphi \simeq \mathbb{1}_X$, there is a htpy $f_t : X \to X$ s.t. $\psi\varphi = f_0$, $\mathbb{1}_X = f_1$ and hence $h(s) = f_s(x_0)$ is a path from $\psi\varphi(x_0) \rightsquigarrow x_0$

Similarly we have a path $g(s) = \varphi \circ f_s(x_0)$ in Y from $\varphi\psi\varphi(x_0) \rightsquigarrow \varphi(x_0)$.

Then $\varphi_*' \beta_h = \beta_g \varphi_*$ is surjective and β_g is an isom

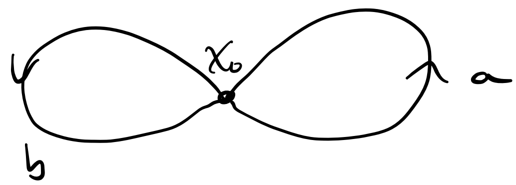
$\Rightarrow \varphi_*$ is surjective.

$$\begin{tikzcd}
\pi_1(X, \psi\varphi(x_0)) \arrow[r, "\varphi_*'", "\text{surjective}"'] & \pi_1(Y, \varphi\psi\varphi(x_0)) \\
\pi_1(X, x_0) \arrow[u, "\beta_h", "\cong"'] \arrow[r, "\varphi_*"'] & \pi_1(Y, \varphi(x_0)) \arrow[u, "\beta_g"', "\cong"]
\end{tikzcd}$$

Toward Seifert – van Kampen theorem...

Free products of groups

eg. Consider $\pi_1(X, x_0)$ where



Believable that $\pi_1(X)$ consists of paths of the form $(n_i, m_i \in \mathbb{Z})$

$a^{n_1} b^{m_1} a^{n_2} b^{m_2} \cdots a^{n_k} b^{m_k}$

(paths are finite...)

We write this group as $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$, the <u>free product</u> of $\mathbb{Z}$ and $\mathbb{Z}$.

<u>defn</u> Let $\{G_\alpha\}_{\alpha \in A}$ be a collection of groups. The free product $\ast_\alpha G_\alpha$ is defined as

- Underlying set: words $g_1 \ldots g_m$ of arbitrary finite length $m \geq 0$ where each $g_i \in G_{\alpha_i}$ for some α_i.

- Identity is the empty word

- group operation is concatenation.
  ("juxtaposition" in Hatcher)
  $(g_1 \cdots g_m)(h_1, \ldots h_n)$
  $= g_1 \cdots g_m h_1 \ldots h_n.$

- inverses:
  $(g_1 \cdots g_m)^{-1} = g_m^{-1} \cdots g_1^{-1}.$

FYI a word is reduced if you've done all possible cancellations / combinations

e.g. • $gg^{-1} \rightsquigarrow 1$

• $g^3 g^2 \rightsquigarrow g^5$

You might recall that $\mathbb{Z} * \mathbb{Z}$ is actually a (finitely generated) free group:

$$\mathbb{Z} * \mathbb{Z} = \langle a, b \mid \text{no relations} \rangle.$$

But observe that $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is not free:

$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a, b \mid a^2 = b^2 = 1 \rangle.$$

Observe that there is a natural inclusion

$$G_\beta \xhookrightarrow[i_\beta]{} \mathop{*}_\alpha G_\alpha$$ as the words of letters <u>only</u> from G_β.

<u>Universal Property</u>:

Given any collection of homs $\varphi_\alpha : G_\alpha \to H$, $\exists!$ $\varphi : \mathop{*}_\alpha G_\alpha \longrightarrow H$ s.t. the diagram commutes $\forall\, \beta \in A$

$$\begin{tikzcd}
\mathop{*}_\alpha G_\alpha \arrow[dr, dashed, "\exists!\,\varphi"] & \\
G_\beta \arrow[u, "i_\beta"] \arrow[r, "\varphi_\beta"'] & H
\end{tikzcd}$$

Suppose $X = \bigcup_\alpha A_\alpha$ where each A_α is path-cntd and contains the basepoint $x_0 \in X$.

For each α, we get homs induced by inclusion

$$j_\alpha : \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0)$$

By the universal property of free products, these extend to a hom

$$\Phi : \mathop{*}_\alpha \pi_1(A_\alpha, x_0) \longrightarrow \pi_1(X, x_0).$$

The Seifert-van Kampen theorem

① tells us when Φ is surjective

② and sometimes, we know what the kernel is

$\Rightarrow$ Under certain conditions, we can compute

$$\pi_1(X, x_0) \cong \ast_\alpha \pi_1(A_\alpha, x_0) / \ker \Phi.$$

<u>thm 1.20</u> (Seifert - Van Kampen Theorem)

If X is the union of path-cntd open sets A_α each containing the basepoint $x_0 \in X$,

① if each intersection $A_\alpha \cap A_\beta$ is path-cntd, then the hom $\Phi : \ast_\alpha \underbrace{\pi_1(A_\alpha)} \to \underbrace{\pi_1(X)}$ is surjective

(basepoint x_0 understood)

② if, in addition to ①, each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-cntd, then $\ker \Phi$ is the normal subgroup N generated by all the elements of the form

$$\underbrace{i_{\alpha\beta}}(\omega)\, \underbrace{i_{\beta\alpha}}(\omega)^{-1} \quad \text{for } \omega \in \pi_1(A_\alpha \cap A_\beta)$$

$i_{\alpha\beta}$ induced by $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$

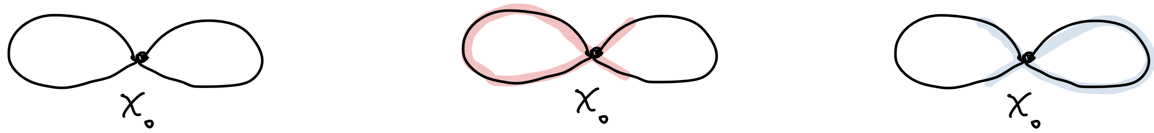
$i_{\beta\alpha}$ induced by $A_\alpha \cap A_\beta \hookrightarrow A_\beta$

& Observe that $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$

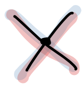
$$\Rightarrow \pi_1(X) \cong \ast_\alpha \pi_1(A_\alpha) / N$$

Some examples before we prove the theorem next week...

eg.



X $\qquad\qquad$ A_1 $\qquad\qquad$ A_2

$A_1 \cap A_2 =$  path cntd ✓.

Triple intersection condition trivially satisfied.

Any $\omega \in \pi_1(A_1 \cap A_2)$ is already trivial

$\Rightarrow \quad \pi_1(X) \cong \pi_1(A_1) * \pi_1(A_2) \cong \mathbb{Z} * \mathbb{Z} = F_2$

free group on 2 generators

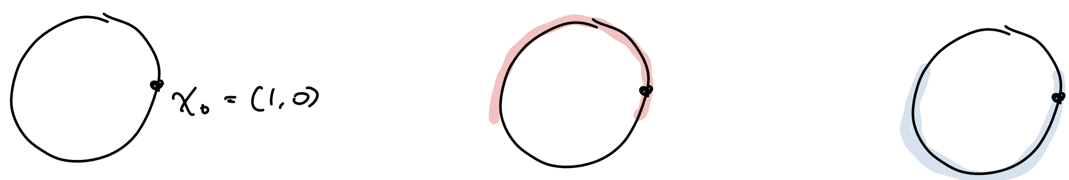
eg. Using a similar argument we can generalize:

If $X = \bigvee_\alpha X_\alpha$ where each X_α is path-cntd

then

$$\pi_1(X) \cong \mathop{\ast}_\alpha \pi_1(X_\alpha).$$

eg. Non example

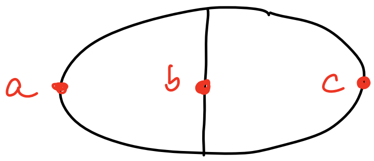


S^1 $\qquad\qquad$ A_1 $\qquad\qquad$ A_2

$A_1 \cap A_2$ is not path cntd.

Indeed, $\pi_1(S^1) \cong \mathbb{Z} \ncong$ trivial group.

eg. Non example



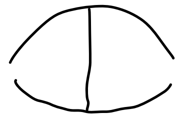
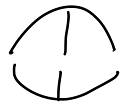
X

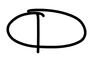
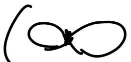
$A_\alpha = X - \{a\}$

$A_\beta = X - \{b\}$

$A_\gamma = X - \{c\}$

$\pi_1(A_j) \cong \pi_1(S^1) \cong \mathbb{Z}$ for $j = \alpha, \beta, \gamma$.

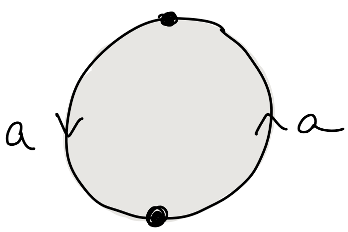
- Each double intersection looks like , path-cntd indeed. (and π_1 trivial)
- However the triple intersection looks like , not path-cntd.

Indeed, $\pi_1($$) \cong \pi_1($$) = \mathbb{Z} * \mathbb{Z}$,

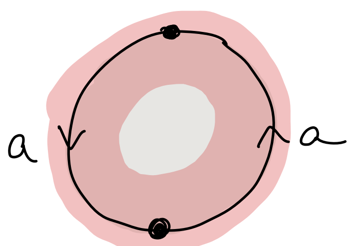
not $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

eg. First non-trivial example extremely instructive.

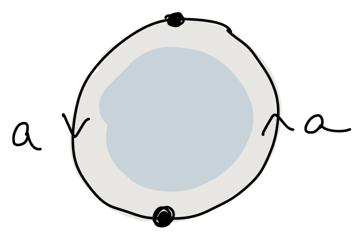
$\mathbb{R}P^2$

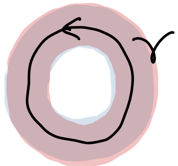


A_1



A_2



$A_1 \cap A_2 =$ , $\pi_1(A_1 \cap A_2)$ is generated by $[\gamma]$

a^2 when written multiplicatively

N is generated by $i_{12}(\gamma)\, i_{21}(\gamma)^{-1} = 2a \cdot 1$.

$\Longrightarrow \pi_1(\mathbb{R}P^2) \cong \mathbb{Z} * 1 / \langle 2a \rangle \cong \mathbb{Z}/2\mathbb{Z}$.