prop. 1.26

(a) If Y is obtained from X by "attaching 2-cells" e^2_α via attaching maps φ_α, then $i_*: \pi_1(X, x_0) \to \pi_1(Y, x_0)$ is surjective & ker i_* is gen'd by elements $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$

change basepoint to x_0.

(b) If Y is obtained from X by attaching n-cells for fixed $n > 2$, then i_* is an isom.

(c) For a path-conn'd cell cpx X, $i: X^2 \hookrightarrow X$ induces an isom

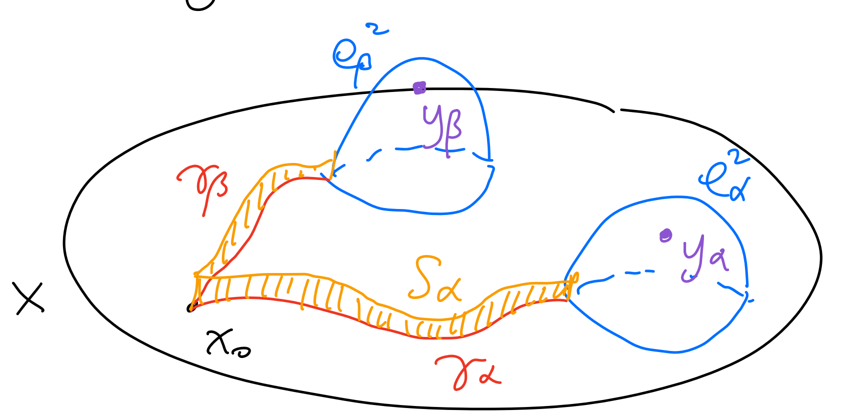
$$i_*: \pi_1(X^2, x_0) \longrightarrow \pi_1(X, x_0).$$

pf. (Sketch)

(a)

(last time)

Setup: expand Y to a slightly larger space Z that def retracts to Y, for sake of S-vk.



S_α = strip along γ_α.

y_α not on S_α.

- $A = Z - \bigcup_\alpha \{y_\alpha\}$ is open, path cnctd,

  def. retracts onto X.

  $B = Z - X$ we arranged so that this would be an open subset of Z. :)

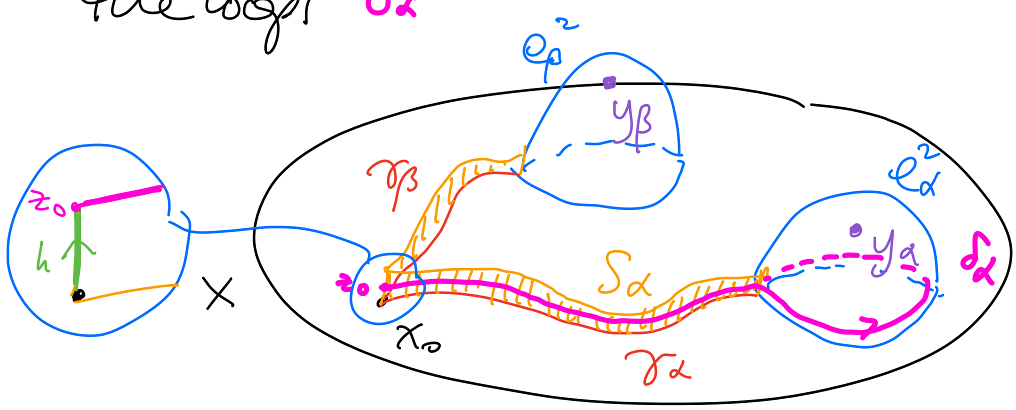
  is contractible.

- S-vK $\Rightarrow$ $\pi_1(Z) \cong \pi_1(A)/K$

  where K = normal subgroup gen'd by image of induced map

  $\pi_1(A \cap B) \longrightarrow \pi_1(A)$. (Pause for group theory)

- IRT check that $\pi_1(A \cap B)$ is generated by the loops δ_α



which are are taken to $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$ under β_h:

Use S-vK again, with open cover

$A_\alpha = A \cap B - \bigcup_{\beta \neq \alpha} e^2_\beta$

end last time

//

(b) Same idea, but use e^n_α, $n>2$ now.

Then A_α deformation retracts to S^{n-1}

$\Rightarrow \pi_1(A_\alpha)=1$.

$\Rightarrow \pi_1(A\cap B)=1$.

(c) If $X=X^n$, then use induction.

If X is not finite dim'l CW cpx, then:

let $[f]\in\pi_1(X,x_0)$.

The based loop has compact image so

lies in X^n for some n. (Fact from App. A.1)

By (b), $f\simeq$ loop in X^2. $\Rightarrow$ $\iota_*: \pi_1(X^2)\to\pi_1(X)$ is surjective.

For injectivity: If f is a (based) loop in X^2

that is null htpc in X via a htpy

$F: I\times I\to X$, then F has cpt

image $\Rightarrow$ is contained in some X^n

(may assume $n>2$).

By (b), $\pi_1(X^2,x_0)\longrightarrow\pi_1(X,x_0)$ is injective

so the f is actually null htpc in X^2 already.

$\blacksquare$

egs. Recall polygonal decomp of M_g, N_g.
(discussion / point-set top course)

eg. $\pi_1(M_g) \cong \langle a_1, b_1, \dots, a_g, b_g \mid \underbrace{[a_1, b_1]}_{\text{review commutators?}} \cdots [a_g, b_g] \rangle$

Cor. If $g \neq h$, then $M_g \not\simeq M_h$ ($\Rightarrow M_g \not\cong M_h$).

pf. Abelianize π_1.

$$Ab(\pi_1(M_g)) \cong \bigoplus_{i=1}^{2g} \mathbb{Z}.$$ //

eg. $\pi_1(N_g) = \langle a_1, b_1, \dots, a_g, b_g \mid a_1^2 \cdots a_g^2 \rangle$

nonorientable, w/ nonorientable genus g.

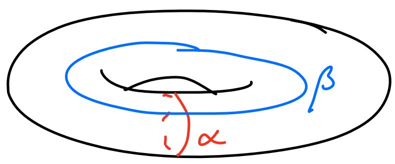
Cor. If $g \neq h$, $N_g \not\simeq N_h$.

Cor For every group G there is a 2-dim'l CW cpx X_G with $\pi_1(X_G) \cong G$.

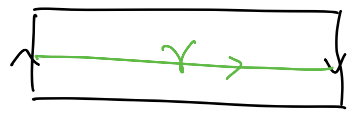
Pf. Use group presentation $G = \langle g_\alpha \mid r_\beta \rangle$
$X^1 = \bigvee_\alpha S^1_\alpha$, attach e^2_β along loops specified by the words r_β. ∎

more examples

eg. What is $\pi_1(X)$ where X is



$\cup$ Mobius band w/ ∂ glued onto β?



S-vK: $\beta \simeq 2\gamma$.

$$\pi_1(X) \cong \langle \alpha, \beta, \gamma \mid \alpha\beta\alpha^{-1}\beta^{-1} = 1,\ \gamma^2 = \beta \rangle$$

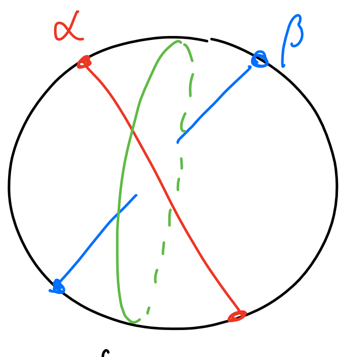
(unclear how to make this group more recognizable)

π_1 elements under homotopy equivalences

(separate topic from above)

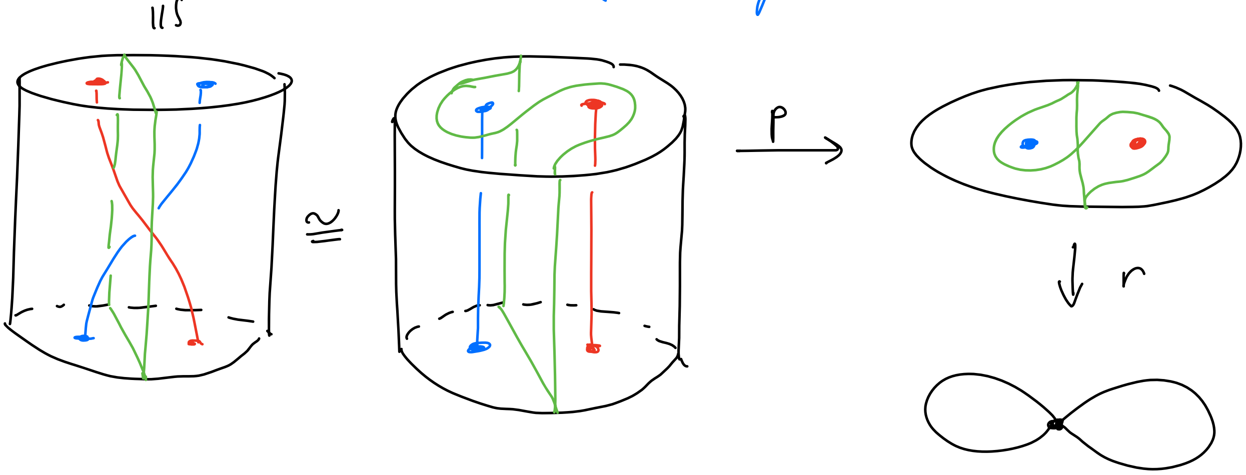
eg. (as inspiration for a few problems)

B^3 (D^3)



Show $[\gamma] \neq 1$ in $\pi_1(D^2 - \alpha - \beta)$.

(both p & r are retractions)



$r_* p_*([\gamma]) = ?$

COVERING SPACES

We calc'd our foundational example $\pi_1(S^1)$ using covering space:

$$\begin{array}{c} \mathbb{R} \\ \downarrow p \\ S^1 \end{array}$$

$\leftarrow$ simply connected.

Can use this method for other cases too b/c of close relationship b/w

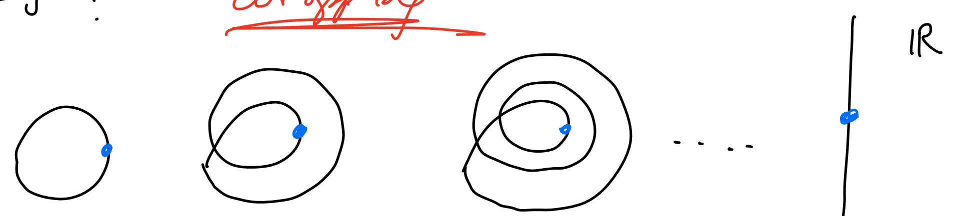
$$\left\{ \begin{array}{c} \text{connected covering spaces} \\ \tilde{X} \text{ of } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{conjugacy classes of} \\ \text{subgroups of } \pi_1(X) \end{array} \right\}$$

basepoint change.

<u>Observations</u> (to conjecture the correct correspondence)

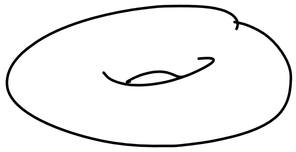
- Can think of any connected covers of $\mathbb{R}$?
- What about S^1? "wrapping"

$\tilde{X}$



p:	$S^1 \xrightarrow{z} S^1$	$S^1 \xrightarrow{z^2} S^1$	$S^1 \xrightarrow{z^3} S^1$	$\mathbb{R} \xrightarrow{\text{mod } 1} S^1$
$p_*(\pi_1(\tilde{X}, \tilde{x}_0))$	$\mathbb{Z}$	$2\mathbb{Z}$	$3\mathbb{Z}$	$0\mathbb{Z} = \langle 0 \rangle$

(sit here a moment; mention deck trans if needed) e.g.



$\pi_1(X) \cong \mathbb{Z}\times\mathbb{Z}$. What are all the connected covers?
(abelian again)

Actual classification theorem is more precise: (Goal)

<u>thm 1.38</u>

X = path cntd, locally path cntd, semilocally simply cntd

Then

$$\left\{\begin{array}{l}\text{basepoint-preserving}\\ \text{isom classes of}\\ \text{path-cntd}\\ \text{covering spaces}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{subgroups of}\\ \pi_1(X, x_0)\end{array}\right\}$$

$$p:(\tilde{X},\tilde{x}_0)\to(X,x_0) \longleftrightarrow p_*(\pi_1(\tilde{X},\tilde{x}_0)).$$

If basepoints ignored,

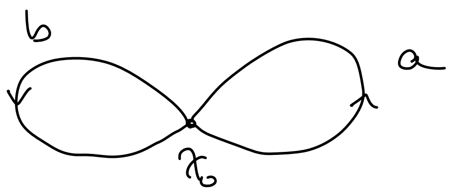
$$\left\{\begin{array}{l}\text{path cntd}\\ p:\tilde{X}\to X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{conjugacy classes}\\ \text{of subgroups}\\ \text{of } \pi_1(X, x_0)\end{array}\right\}.$$

$\Rightarrow$ if $\pi_1(X,x_0)$ is abelian, basepoints "don't matter"

defn/
<u>Cor.</u> Every such space has a simply-cntd cover $\tilde{X}$, called the <u>universal cover</u> (because it covers all other covers) corresponding to $\langle 1\rangle \in \pi_1(X,x_0)$.

Today: get a sense of what's happening.

Consider $X = S^1 \vee S^1$:



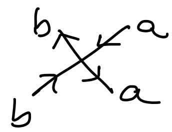
$\pi_1(X, x_0) = \langle a, b \rangle \cong F_2$.

To construct covers corr. to subgroups, we instead "cographs"

temp. name, in Hatcher

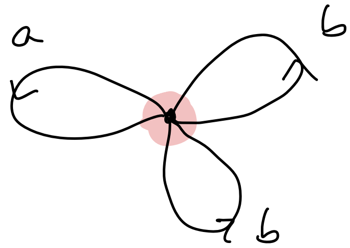
Observe: Covering spaces of X are "2-oriented graphs"

i.e. locally

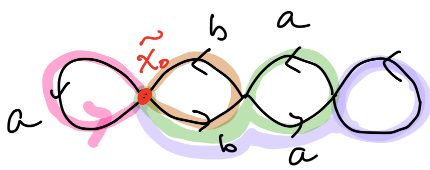


at vertices

non eg.



eg.



ba^2b $\quad b^2$

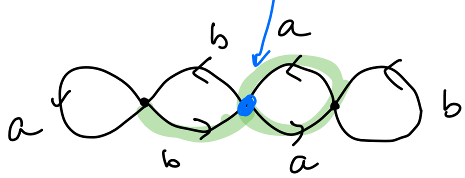
$(ba^2b)(b^2)^{-1}$

$\langle a, b^2, ba^2b^{-1}, bab\,a^{-1}b^{-1} \rangle$

a different lift $\tilde{x}_0$

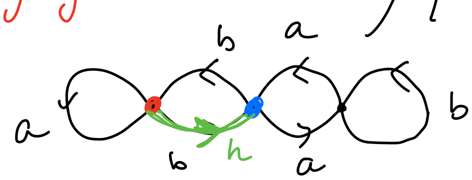
Conjugate by "b"

β_h

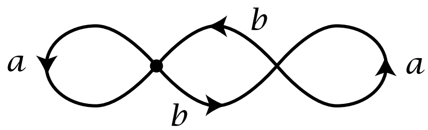
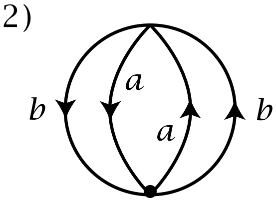
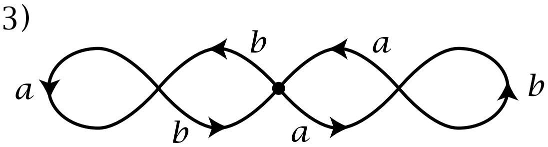
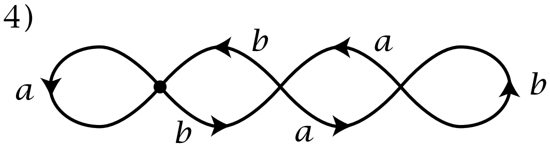
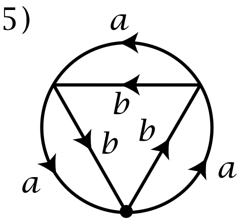
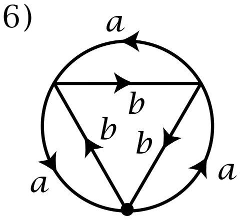
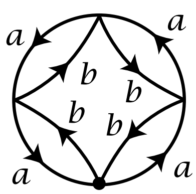
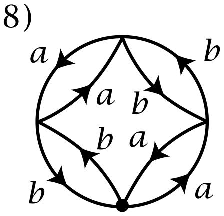
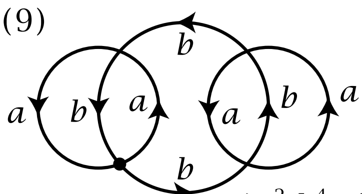
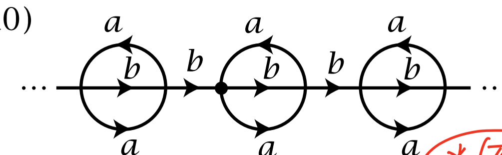
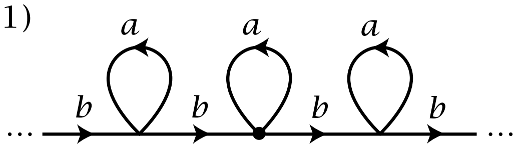
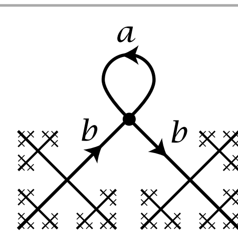
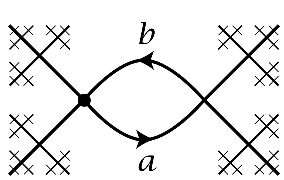
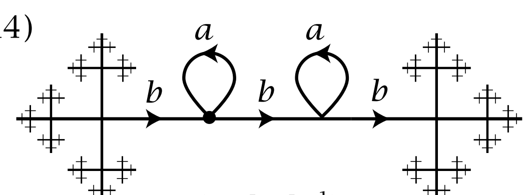


$\langle b^2, bab^{-1}, a^2, aba^{-1} \rangle$

These two are conjugate! Apply β_h along h:



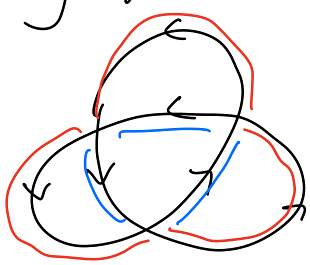
Some Covering Spaces of $S^1 \vee S^1$

(1)  $\langle a, b^2, bab^{-1} \rangle$	(2)  $\langle a^2, b^2, ab \rangle$
(3)  $\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$	(4)  $\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$
(5)  $\langle a^3, b^3, ab^{-1}, b^{-1}a \rangle$	(6)  $\langle a^3, b^3, ab, ba \rangle$
(7)  $\langle a^4, b^4, ab, ba, a^2b^2 \rangle$	(8)  $\langle a^2, b^2, (ab)^2, (ba)^2, ab^2a \rangle$
(9)  $\langle a^2, b^4, ab, ba^2b^{-1}, bab^{-2} \rangle$	(10)  $*_{n \in \mathbb{Z}} (\mathbb{Z} * \mathbb{Z})$ $\langle b^{2n}ab^{-2n-1}, b^{2n+1}ab^{-2n} \mid n \in \mathbb{Z} \rangle$
(11)  $\langle b^n a b^{-n} \mid n \in \mathbb{Z} \rangle$ $*_{n \in \mathbb{Z}} \mathbb{Z}$	(12)  $\langle a \rangle$
(13)  $\langle ab \rangle$	(14)  $\langle a, bab^{-1} \rangle$

Rmk F_2 contains F_k for all $k \in \mathbb{N} \cup$ "countable ∞" !
(see #11)

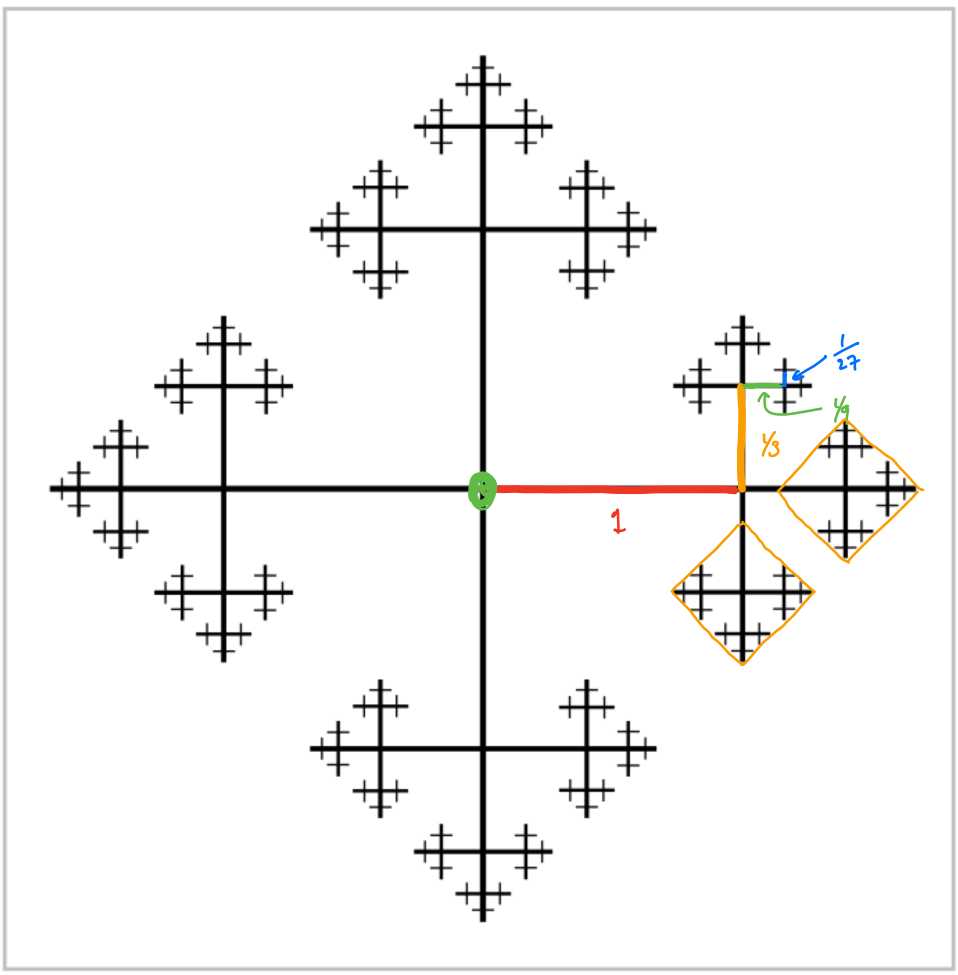
Rmk In particular, every knot diagram gives a covering space:

HW



Universal Cover of $S^1 \vee S^2$:

"unwrap completely":



$\lambda = 1/3$

$(0 < \lambda < 1/2)$

Lifting Properties

Recall A lift of $f: Y \to X$ is a map $\tilde{f}: Y \to \tilde{X}$ s.t. $p\tilde{f} = f$:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"'] \arrow[ur, "\tilde{f}"] & X \end{tikzcd}$$

(thm 1.7 part (c))

Prop 1.30 (Homotopy Lifting Property)

aka covering homotopy property

Given a covering space $p: \tilde{X} \to X$,

a htpy $f_t: Y \to X$, and

a map $\tilde{f}_0: Y \to \tilde{X}$ lifting f_0,

there exists a unique htpy

$\tilde{f}_t: Y \to \tilde{X}$ of $\tilde{f}_0$

that lifts f_t.

$$\begin{tikzcd} Y \times \{0\} \arrow[r, hook] \arrow[rr, bend left, "\tilde{f}_0"] & Y \times I \arrow[r, "\{\tilde{f}_t\}", "\exists!"'] \arrow[d, phantom] & \tilde{X} \arrow[d, "p"] \\ & Y \times I \arrow[r, "\{f_t\}"'] & X \end{tikzcd}$$

(This was actually part (c) of Thm 1.7, already proven.)

Rmk.

- We proved this in Week 3 as claim Ⓒ from the pf. of theorem 1.7. ($\pi_1(S^1) \cong \mathbb{Z}$).
- Think of this as a special property of covering maps $p: \tilde{X} \to X$.

① The case where $Y = *$ is called the "path-lifting property":

> For every $f: I \to X$ & choice $\tilde{x}_0 = \tilde{f}(0)$, $\exists!$ path $\tilde{f}: I \to \tilde{X}$ starting at $\tilde{x}_0$ lifting f.

* In particular, every lift of a constant path is constant ⊛

② The case where $Y = I$ shows every htpy f_t of f_0 lifts to a htpy $\tilde{f}_t$ for each lift $\tilde{f}_0$ of f_0.

* In particular if f_t is a htpy of paths, $\tilde{f}_t(1)$ traces out a constant path

We can use HLP to prove part of our goal thm:

prop. 1.31

(a) $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ induced by a covering space $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is injective.

(b) The image subgroup consists of htpy classes of loops in X based at x_0, whose (unique) lifts to $\tilde{X}$ (starting at $\tilde{x}_0$) are loops.

pf.

(a) Let $[\tilde{f}_0] \in \ker p_*$ (WTS $\tilde{f}_0 \simeq c_{\tilde{x}_0}$ const)

$\Rightarrow [f_0] = p_*[\tilde{f}_0] \simeq [c_{x_0}]$ so

$\exists$ htpy f_t from f_0 to $f_1 = c_{x_0}$.

This lifts to a htpy from $\tilde{f}_0$ to $\tilde{f}_1 = \tilde{c}_{x_0}$

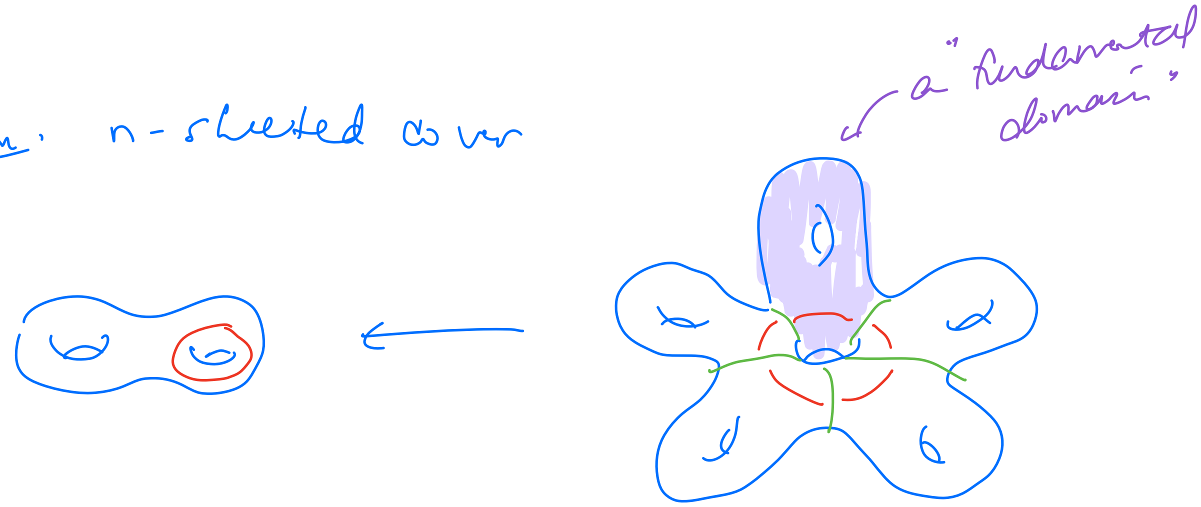
by above comment ✱

(b) Clearly loops @ x_0 lifting to loops at $\tilde{x}_0$ are in image.

Conversely, if a loop $g \simeq$ loop f w/ such a lift,

by htpy lifting property, g also has such a lift. ∎

defn. n-sheeted cover



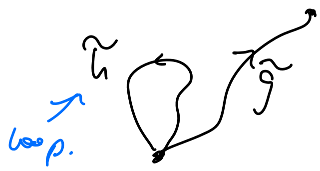
prop 1.32 # sheets of $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ $\underbrace{\phantom{(\tilde{X}, \tilde{x}_0)}}$ path cntd

$= $ index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.

Pf. let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Consider # cosets of H.

loop g based at x_0, lift $\tilde{g}$ starting at $\tilde{x}_0$.

$[h] \in H$. Consider $\tilde{h} \cdot \tilde{g}$:



$\Rightarrow$ Define function

$$\left\{ \text{right cosets of } H \right\} \xrightarrow{\Phi} \left\{ p^{-1}(x_0) \right\}$$

$$H[g] \longmapsto \tilde{g}(1).$$

* Check well-defined ① $g \in [g]$ ② $h[g] \in H[g]$

Φ is surjective: Clear as $\tilde{X}$ is path-cntd; choose $\tilde{g}$ from $\tilde{x}_0$ to another lift; this projects to a loop $g = p \circ \tilde{g}$

Φ is injective: $\Phi(H[g_1]) = \Phi(H[g_2])$

$\Rightarrow$ $g_1 \cdot \overline{g_2}$ lifts to a loop $\tilde{g}_1 \cdot \overline{\tilde{g}_2}$ at $\tilde{x}_0$.

$\Rightarrow$ $[g_1][g_2]^{-1} \in H \Rightarrow H[g_1] = H[g_2]$.

$\blacksquare$