Summary of Friday results:

Ⓐ Homotopy lifting property (of covering maps)

Relating maps from $Y \to X$ to maps $Y \to \tilde{X}$:

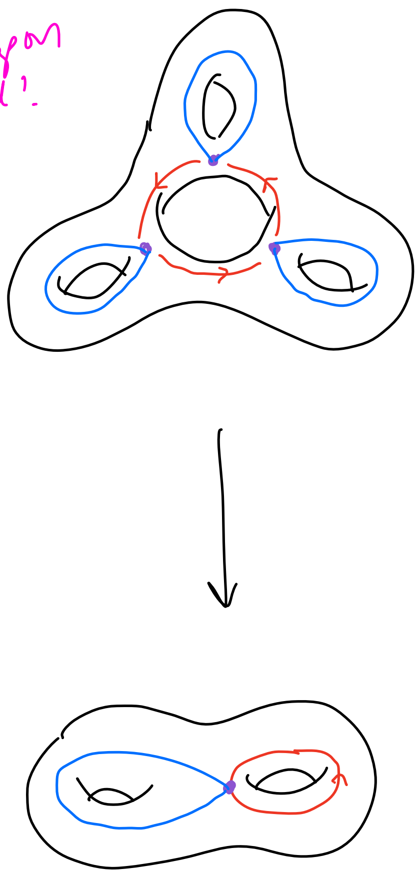
$$\begin{tikzcd} & & \tilde{X} \arrow[d, "p"] \\ Y \times \{0\} \arrow[r, hook] \arrow[urr, "\tilde{f}_0"] & Y \times I \arrow[r, "\{f_t\}"'] \arrow[ur, dashed, "\exists! \ \{\tilde{f}_t\}"] & X \end{tikzcd}$$

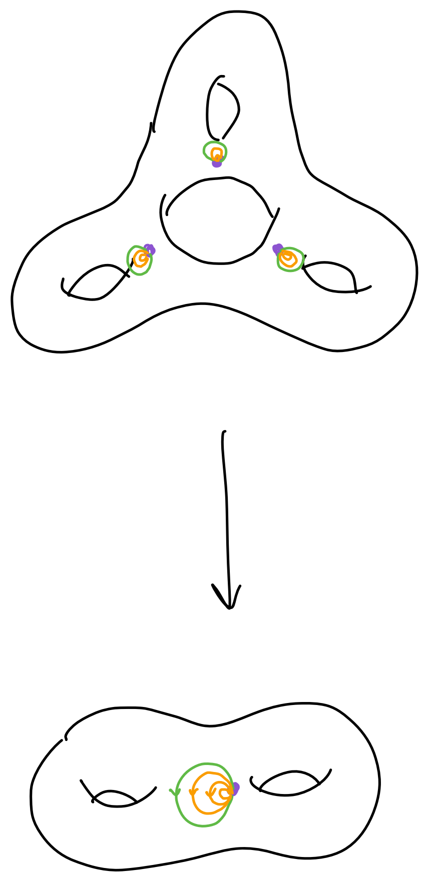
In particular:

① path lifting

② lifting htpy of paths

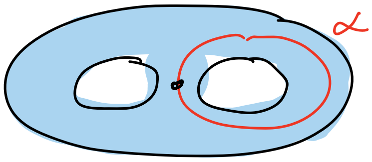
* keep on board!





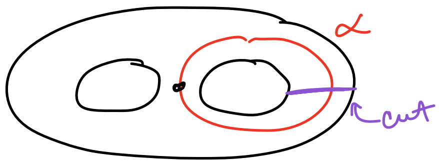
Aside: Building (small) covers

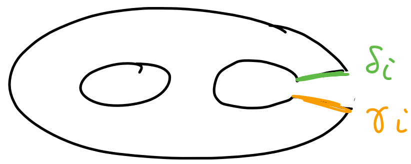
eg. Consider

$X =$ 

$= D^2 \setminus$ interior of 2 D^2s

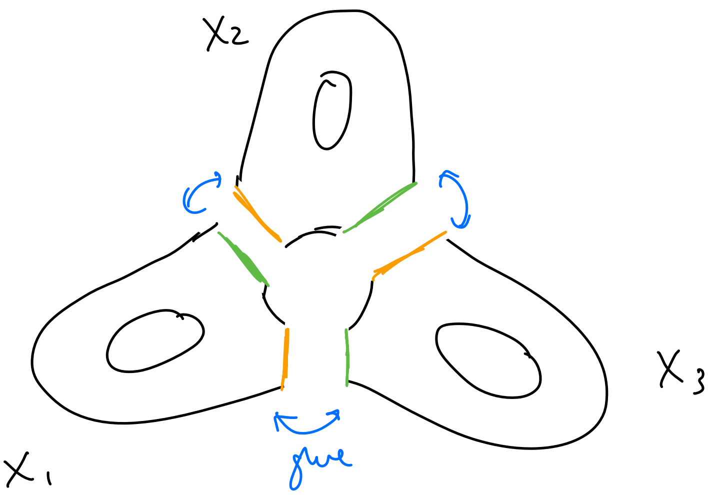
Want to unwrap α by a factor of 3:



Take 3 copies $i \in \{0,1,2\}$ $\quad X_i =$ 

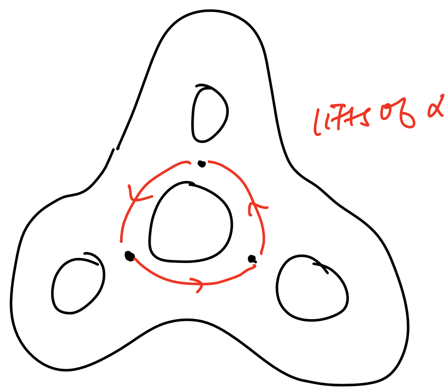
/ δ_i glued to γ_i

Glue δ_i to γ_{i+1} (mod 3)



$=$

$\tilde{X}$



(B) <u>prop 1.31, 1.32</u> $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ covering space

(a) p_* is injective

(b) image(p_*) = htpy classes of based loops that lift to (based) <u>loops</u> (see figure above)

(c) index of img(p_*) = # sheets of p ← cardinality

more precisely, $\exists$ bijection:

$$\left\{ \begin{matrix} \text{right cosets of} \\ \text{img } p_* \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{preimages of} \\ x_0 \end{matrix} \right\}$$

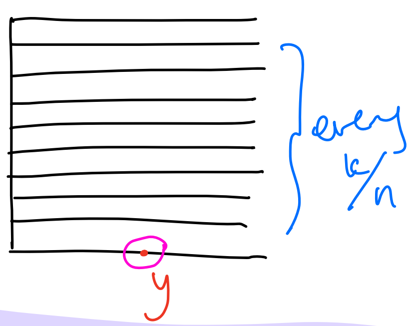
~

Continue: What about lifts of general maps (not just homotopies)?

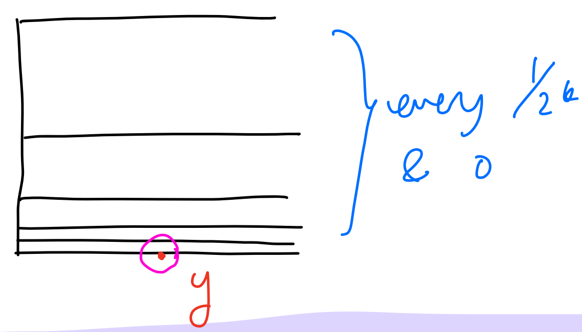
- htpies are special because...

<u>defn.</u> A space Y is <u>locally path-cntd</u> if $\forall\, y \in Y$ and each nbhd U of y, $\exists$ open nbhd $V \subset U$ of y that is path cntd.

<u>example</u>



<u>non example</u>



[Existence of lifts]

Prop 1.33 (Lifting Criterion) for general maps

Given $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ covering space

$f : (Y, y_0) \to (X, x_0)$

with Y path-cntd & locally path cntd,

then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ exists

iff $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & (X, x_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(\tilde{X}, \tilde{x}_0) \arrow[d, "p_*"] \\
\pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(X, x_0)
\end{tikzcd}$$

pf. $\boxed{\Rightarrow}$ clear from right diagram: $f_* = p_* \tilde{f}_*$

$\boxed{\Leftarrow}$

(1) Use path-cntdness to define $\tilde{f}$

Let $y \in Y$.

Pick path $y_0 \overset{\gamma}{\rightsquigarrow} y$.

Then $f\gamma$ is a path in X starting at x_0.

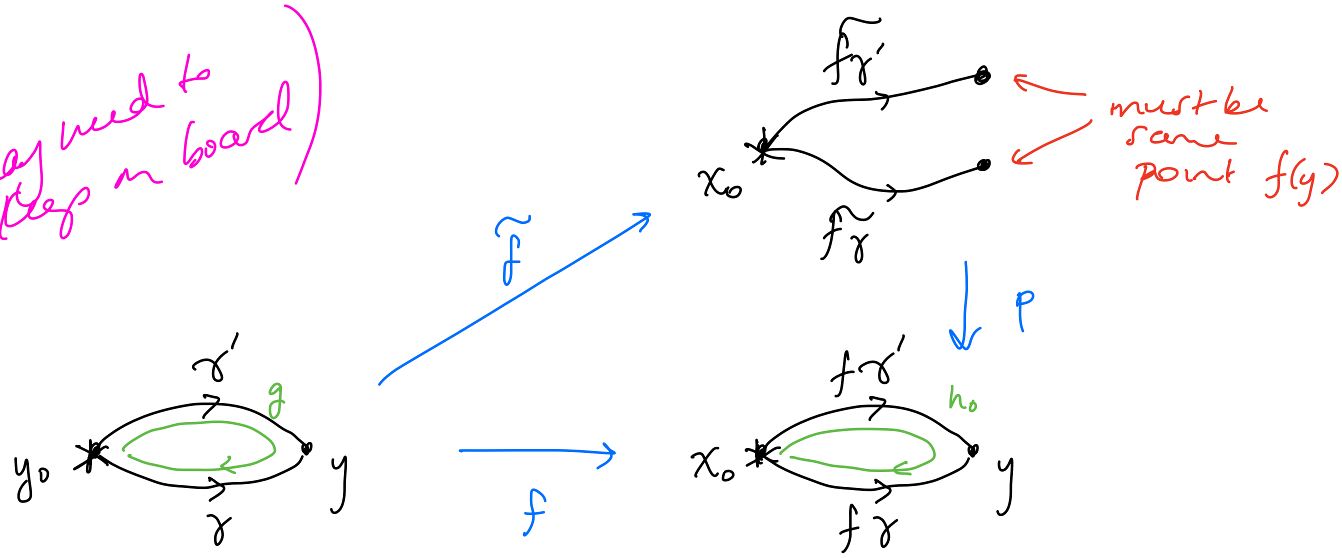
$\overset{HLP}{\Longrightarrow}$ $\exists!$ lift $\widetilde{f\gamma}$ starting at $\tilde{x}_0$.

Define $\tilde{f}(y) = \widetilde{f\gamma}(1)$.

we made a choice: path γ.

Need to show $\tilde{f}(y)$ is well-defined.

(may need to keep on board)



If you chose instead $y_0 \underset{\gamma'}{\rightsquigarrow} y$,

$g := \gamma'\bar{\gamma}$ is a loop based at y_0

$\Rightarrow [g] \in \pi_1(Y, y_0)$

$\Rightarrow h_0 := (f\gamma') \cdot \overline{(f\gamma)}$ is a loop based at x_0

By assumption $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$\Rightarrow [h_0] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$\Rightarrow \exists\ h_1 \in [h_0]$ s.t. $\tilde{h}_1$ is a loop in $\tilde{X}$ based at $\tilde{x}_0$

Since h_1 is a loop, h_0 must also be.

Common argument $\Rightarrow$ h_0 is in fact a loop.

By uniqueness (HLP),

$\tilde{h}_0|_{[0,0.5]}$ is $\widetilde{f\gamma'}$ (twice as fast) and

$\tilde{h}_0|_{[0.5,1]}$ is $\widetilde{\overline{f\gamma}}$ (twice as fast)

$\Rightarrow \widetilde{f\gamma'}(1) = \widetilde{\overline{f\gamma}}(0) = f\gamma(1) = h_0(0.5)$.

(hence the choice of γ does not matter, and $\tilde{f}$ is well defined)

② Use local path-connectedness to show $\tilde{f}$ is continuous

Let U be an evenly covered (open) nbhd of $f(y)$

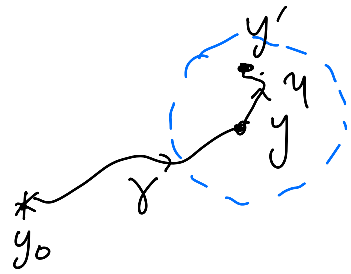
i.e. $f(y) \subset U \subset X$, U has lift $\tilde{U}$ s.t. $p: \tilde{U} \to U$ is a homeo.

Choose a path-cnted open nbhd V of y s.t. $f(V) \subset U$

i.e. f cont $\Rightarrow$ $f^{-1}(U)$ open; local path cntdness of Y $\Rightarrow$ $\exists$ path cnted $V \subset f^{-1}(U)$.

(WTS $\tilde{f}|_V$ is continuous)

For $y' \in V$, choose a fixed path $y_0 \overset{\gamma}{\rightsquigarrow} y$ and path $y \overset{\eta}{\rightsquigarrow} y'$ $(\eta \subset V)$.



Then:

$f \circ (\gamma \cdot \eta) = (f\gamma) \cdot (f\eta)$ has unique lift $(\widetilde{f\gamma}) \cdot (\widetilde{f\eta})$

where $\widetilde{f\circ\eta} = p^{-1} \circ f \circ \eta$ (where p^{-1} is the local inverse.)

$\Rightarrow$ ① $\tilde{f}(V) \subset \tilde{U}$ (Image of p^{-1}) and

② $\tilde{f}|_V(y') = \tilde{f}|_V(\eta(1))$

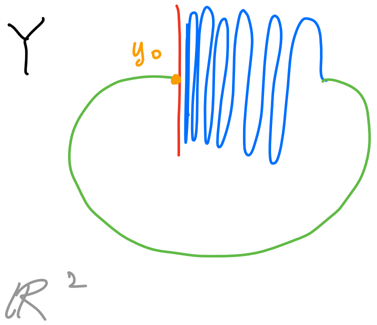
$= p^{-1} \circ f \circ \eta(1) = p^{-1} \circ f(y')$

$\Rightarrow \tilde{f}|_V = p^{-1} f$ which is continuous.

$\Rightarrow \tilde{f}$ is continuous at y. ∎

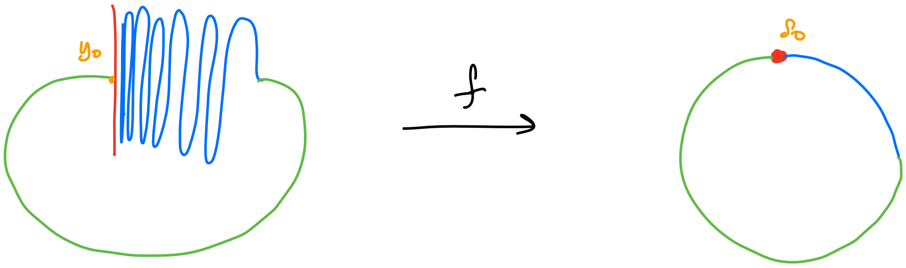
The lifting criterion can indeed fail when Y is not locally path cntd:

eg The quasi-circle : Y = union of 3 pieces :



- $[-1,1]$ on y-axis
- portion of $y = \sin(\frac{1}{x})$ to for $x \in (0, 1/\pi]$
- an arc from $(\frac{1}{\pi}, 0)$ to $(0,0)$ as shown.

- not locally path cntd: consider points in red.
- $\exists$ quotient map $f : Y \to S^1$ that sends all of red to a point



* mentally check this is continuous

- There is a covering space $p : \mathbb{R} \to S^1$
  $\pi_1(Y, y_0) = 1 \subset p_*(\pi_1(\mathbb{R}, 0)) = 1 \subset \pi_1(S^1, s_0) \cong \mathbb{Z}$.
- But f does not lift to $\tilde{f} : Y \to \mathbb{R}$

If $\tilde{f}$ were a lift, by continuity y_0 would have to be sent to both n and $n+1$ in $\mathbb{R}$ (lifts of s_0).

[uniqueness of lifts]

prop 1.34 (Unique lifting property)

can also view as a criterion.

Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$, if two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ agree at one point of Y, and Y is connected, then $\tilde{f}_1$ and $\tilde{f}_2$ agree on all of Y.

ie. given

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd}$$

covering space

connected

if $\tilde{f}$ exists, it is unique.

Pf.

Let $y \in Y$, WTS $\tilde{f}_1(y) = \tilde{f}_2(y)$.

U an evenly covered open nbhd of $f(y)$, in X

$\tilde{U}_1, \tilde{U}_2$ the sheets containing $\tilde{f}_1(y), \tilde{f}_2(y)$

$\tilde{f}_i$ cts $\Rightarrow \exists$ nbhd N_i of y s.t. $\tilde{f}_i(N_i) \subset \tilde{U}_i$

Take $N = N_1 \cap N_2$.

- If $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ then $\tilde{U}_1 \cap \tilde{U}_2$

  $\Rightarrow \quad \tilde{f}_1(y') \neq \tilde{f}_2(y') \quad \forall\, y' \in N.$

  $\Rightarrow \{ y \in Y \mid \tilde{f}_1(y) \neq \tilde{f}_2(y) \}$ is open in Y

- If $\tilde{f}_1(y) = \tilde{f}_2(y)$ then $\tilde{U}_1 = \tilde{U}_2$

  $\Rightarrow \tilde{f}_1 = \tilde{f}_2$ on N (since $p\tilde{f}_1 = p\tilde{f}_2$ and p is injective on N)

  $\Rightarrow \{ y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y) \}$ is open in Y.

  (& also closed, from above)

$\Rightarrow \{ y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y) \}$ = either Y or ϕ. ∎

Now toward proving classification of covering spaces.

We restrict to locally path cntd X.

- locally path cntd $\Rightarrow$ path components = components

So X path cntd $\Leftrightarrow$ X connected.

- X loc path cntd $\Rightarrow$ $\tilde{X}$ is loc path cntd

So $\tilde{X}$ path-cntd $\Leftrightarrow$ $\tilde{X}$ connected.

$\Rightarrow$ we can stop saying "path" as long as X is locally path cntd.

1st question: When does X have a simply connected covering space?

Necessary condition:

dfn. X is semilocally simply-connected if every point $x \in X$ has a nbhd U s.t. the inclusion-induced hom

$$i_* : \pi_1(U, x) \longrightarrow \pi_1(X, x)$$

is trivial.

__Claim__ If X has a simply cntd $\tilde{X}$, then X must be semilocally simply cntd.

Pf. $p : \tilde{X} \to X$ universal cover.

$x \in X$ has evenly covered nbhd U.

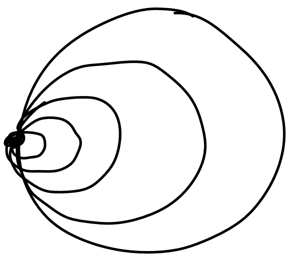
Pick any lift $\tilde{x} \in \tilde{U}$.

Each loop γ in U lifts to a null htpc loop in $\tilde{U}$ (since $\pi_1(\tilde{X}) = 1$)

$\Rightarrow$ $p \circ \gamma$ is null htpc too. □

e.g. non semilocally simply cntd space: (Example 1.25)

x_0



"Shrinking wedge of circles"

Consider the "wedge" point. Any nbhd U of x_0 contains a fractal copy of X

$\Rightarrow$ img of i_* contains $\ast^{\infty} \mathbb{Z}$!

Aside

Eg 1.25 Circle n "C_n" has radius $1/n$, $n = 1, 2, 3 \ldots$

retractions $r_n : X \to C_n$ collapses all other pts to basepoint

$\rightsquigarrow$ ∞ surjections $f_n : \pi_1(X) \to \pi_1(C_n) \cong \mathbb{Z}$

$\Rightarrow$ get hom $f : \pi_1(X) \to \prod^{\infty} \mathbb{Z}$ $\leftarrow$ uncountable (Cantor diag argument)

Surjective: Construct loop from any sequence (k_n)

wraps k_n times around C_n in $[1-1/n, 1-1/n+1]$

check continuous at time 1:

every nbhd of x_0 contains all but finitely many of the C_n ✓.

$\Rightarrow$ $\pi_1(X)$ is uncountable.

On the other hand $\pi_1(\bigvee^{\infty} S^1)$

is countably generated $\Rightarrow$ countable.