Recall from last week:

- Semilocally simply connected:

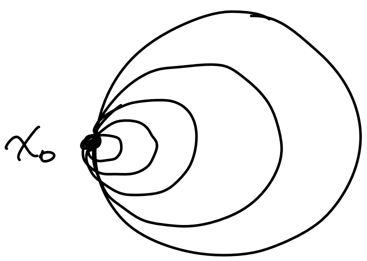
  $\forall\, x \in X$, $\exists$ nbhd $U \ni x$ s.t.

  $i_* : \pi_1(U, x) \longrightarrow \pi_1(X, x)$ is trivial.

- If X has a simply cntd $\tilde{X}$, then

  X must be semilocally simply cntd.

eg. non semilocally simply cntd space: (Example 1.25)



now we will show $\pi_1(X)$ is in fact uncountable!

Circle n "C_n" has radius $1/n$, $n = 1, 2, 3 \ldots$

retractions $r_n : X \to C_n$ collapses all other pts to basepoint

$\rightsquigarrow$ ∞ surjections $f_n : \pi_1(X) \to \pi_1(C_n) \cong \mathbb{Z}$

$\Rightarrow$ get hom $\rho : \pi_1(X) \longrightarrow \prod^{\infty} \mathbb{Z}$ $\leftarrow$ uncountable (Cantor diag argument)

Surjective: Construct loop from any sequence (k_n)

wraps k_n times around C_n in $[1 - 1/n, 1 - 1/n+1]$

check continuous at time 1:

every nbhd of x_0 contains all but finitely many of the C_n ✓.

$\Rightarrow \pi_1(X)$ is uncountable.

On the other hand $\pi_1(\bigvee^{\infty} S^1)$ is countably generated $\Rightarrow$ countable.

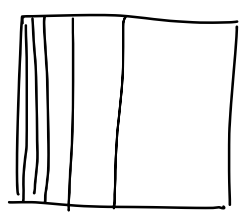
Fact: CW cpxs are even better — they are locally contractible

$\Rightarrow$ $\forall x, \forall U \ni x, \exists$ nbhd $V \subset U$ of x s.t. V is contractible. } "weakly" locally contractible

$\Rightarrow$ shrinking wedge of circles CANNOT be realized as a CW cpx!

eg. Observe that CX is semilocally s.c but not "locally simply cnted".

eg. path cntd, not locally path cntd, semilocally simply cntd (actually contractible)

CX where $X =$ 

Construction Suppose X is path-cntd PC

& locally path cntd LPC

& semi locally simply cntd. SLSC

write on how common this construction is for universal bdl.

- How to construct a simply cntd covering space $\tilde{X}$ (the universal cover):

$$\tilde{X} = \{ [\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0 \}$$

- makes sense since $[\gamma] \to \gamma(1)$ is a well-defined function by HLP. (assuming $\tilde{X}$ exists already).

- Now need to define a topology on this space:

main concern: need to have $p: \tilde{X} \to X$ be a covering map.

Observation

- let $\mathcal{U} = \{$ path cntd opens $U \subset X$ s.t. $\pi_1(U) \to \pi_1(X)$ is trivial $\}$.
- If V path cntd, $V \subset U$, $U \in \mathcal{U}$, then $V \in \mathcal{U}$ too.

SLSC

locally path cntd.

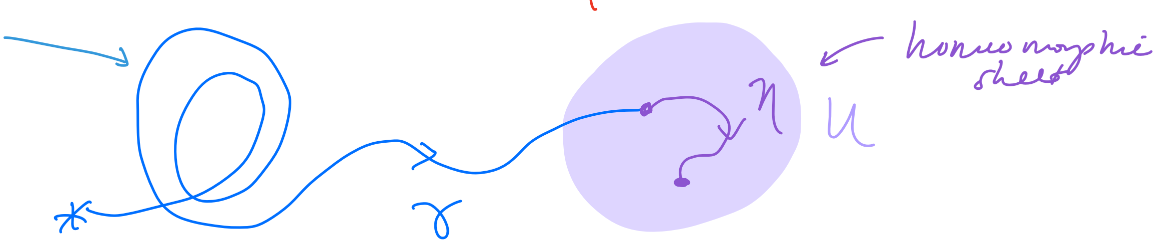
- $\Rightarrow \mathcal{U}$ is a basis for the topology on X (for X LPC & SLSC). ← think through this.

- Given $U \in \mathcal{U}$, path $[\gamma]$ in X starting at x_0 and ending at a pt in U,

let

$$U_{[\gamma]} = \{ [\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1) \}$$

records which "lift" $\tilde{U} = U_{[\gamma]}$



- check: $U_{[\gamma]}$ depends only on the htpy class of γ, indeed.

- $p: U_{[\gamma]} \to U$ surjective since U path cntd
- p injective since different η from $\gamma(1) \rightsquigarrow x \in U$ $\quad (x = \eta(1))$ are all htpc in X, since $\pi_1(U) \to \pi_1(X)$ is trivial.

So $p: U_{[\gamma]} \to U$ is a bijection (of sets)

- Observe $[\gamma'] \in U_{[\gamma]} \implies U_{[\gamma]} = U_{[\gamma']}$.

$\gamma' = \gamma \cdot \eta \implies$ elts of $U_{[\gamma']}$ are $[\gamma \cdot \underline{\eta \cdot \mu}] \in U_{[\gamma]}$

& If $[\gamma \cdot \mu] \in U_{[\gamma]}$ then

$$[\gamma \cdot \mu] = [\underbrace{\gamma \cdot \eta}_{\gamma'} \cdot \bar{\eta} \cdot \mu] \in U_{[\gamma']}.$$

Thus the sets $\{U_{[\gamma]}\}$ form a basis for the topology on $\tilde{X}$:

Given $U_{[\gamma]}$, $V_{[\gamma']}$, $[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$,

then $U_{[\gamma]} = U_{[\gamma'']}$ & $V_{[\gamma']} = V_{[\gamma'']}$.

$\Rightarrow$ If $W \in \mathcal{U}$ is contained in $U \cap V$, and contains $\gamma''(1)$, then

$[\gamma''] \in W_{[\gamma'']} \subset U_{[\gamma'']} \cap V_{[\gamma'']}$.

- $p: U_{[\gamma]} \longrightarrow U$ is a homeo (already showed bijection)

  (and hence $p: \tilde{X} \longrightarrow X$ is continuous)

- $p(V_{[\gamma']}) = V$ $\qquad p^{-1}: U \longrightarrow U_{[\gamma]}$ cont.

- $p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ $\qquad p: U_{[\gamma]} \to U$ cont.

  for $[\gamma'] \in U_{[\gamma]}$ with endpoint in V

since

$$\begin{array}{c} V_{[\gamma']} \subset U_{[\gamma']} = U_{[\gamma]} \\ \downarrow p \\ V \end{array}$$

Friday

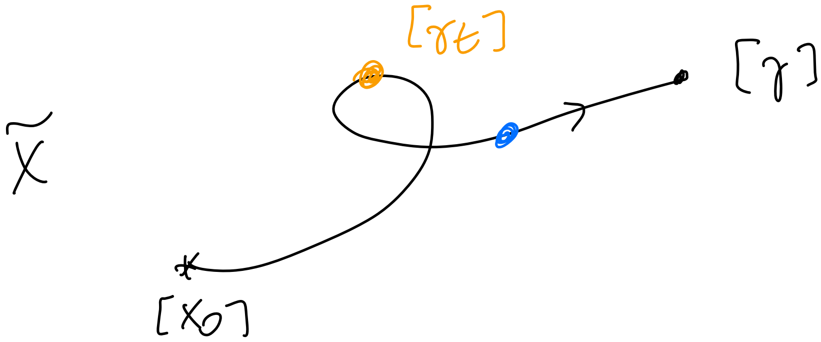
- IRTS $\tilde{X}$ is simply connected.

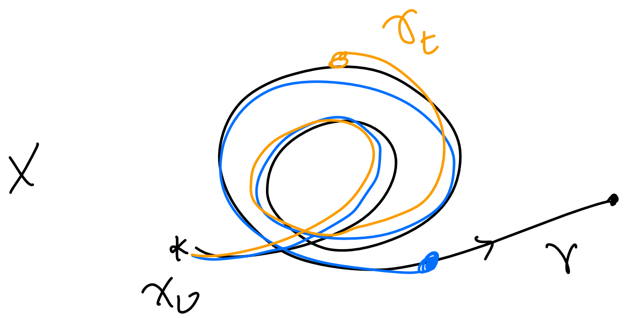
① clear that $\tilde{X}$ is path cntd.

let γ_t = γ on $[0,t]$, stationary on $[t,1]$ (@ $\gamma(t)$)

$t \mapsto [\gamma_t]$ is a path from $[x_0]$ to $[\gamma]$ lifting γ.

const





② Since p_* is injective, to show $\pi_1(\tilde{X},[x_0])=1$, IRTS $p_*(\pi_1(\tilde{X},[x_0]))=1$.

If the path $t \mapsto [\gamma_t]$ that lifts γ (starting at $[x_0]$) is a loop, then $[\gamma_1]=[x_0]$.

But $\gamma_1 = \gamma \Rightarrow [\gamma]=[x_0]$

$\Rightarrow \gamma$ is null htpc.

We now see why we need all 3 conditions:

PC, LPC, SLSC

Next: main thm: these conditions are enough for a robust classification of covering spaces.

prop 1.37 Suppose X is PC, LPC, SLSC.

Then $\forall H \leq \pi_1(X, x_0)$, $\exists$ covering space $p: X_H \to X$ s.t. $p_*(\pi_1(X_H, \tilde{x}_0)) = H$ for a suitably chosen basepoint $\tilde{x}_0 \in X_H$.

Pf

By construction, taking quotient of $\tilde{X}$.

- For $[\gamma], [\gamma'] \in \tilde{X}$, define an equiv relation,

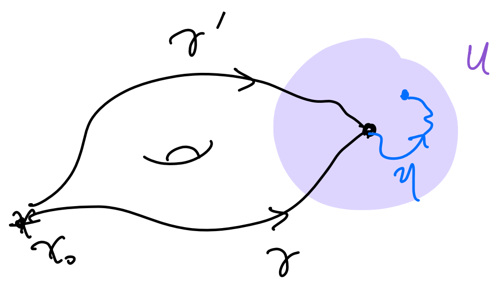
$$[\gamma] \sim [\gamma'] \quad \text{if} \quad \gamma(1) = \gamma'(1)$$
$$\& \quad [\gamma \cdot \bar{\gamma}'] \in H.$$

($\hookrightarrow$ check this is an equiv relation.)

- Define $X_H = \tilde{X}/\sim$.

- Note that if $\gamma(1) = \gamma'(1)$, then

$$[\gamma] \sim [\gamma'] \quad \underline{\text{iff}} \quad [\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$$



$\Rightarrow$ If $[\beta] \in U_{[\gamma]}$ and $[\beta'] \in U_{[\gamma']}$ are identified in X_H ($[\beta] \sim [\beta']$) then in fact $U_{[\gamma]}$ is identified with $U_{[\gamma']}$.

$\Rightarrow$ the natural map $X_H \to X$ induced by $[\gamma] \mapsto \gamma(1)$ is a covering space.

- <u>Basepoint</u>: choose $\tilde{x}_0 \in X_H$ to be the equiv class of c_{x_0}.

  Then $p_*(\pi_1(X_H, \tilde{x}_0)) = H$:

  If a loop γ in X (based at x_0) its lift $\tilde{\gamma}$ is a path $[c_{x_0}] \rightsquigarrow [\gamma]$.

  So $\tilde{\gamma}$ is a loop iff $[\gamma] \sim [c]$, ie if $[\gamma] \in H$. ▪

We have existence. NTS uniqueness.

To be clear, up to <u>isomorphism</u> of covering spaces:

<u>defn</u> An <u>isomorphism</u> from $p_1 : \tilde{X}_1 \to X$ to $p_2 : \tilde{X}_2 \to X$ is a homeo $f : \tilde{X}_1 \to \tilde{X}_2$ s.t. $p_1 = p_2 f$:

$$\begin{tikzcd} X_1 \arrow[rr, "f", "\cong"'] \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

(Hence f^{-1} is also an isom)

Prop. 1.37 Suppose X is PC, LPC, SLSC.

Two path-connected covering spaces

$$p_1 : \tilde{X}_1 \longrightarrow X \qquad \text{and} \qquad p_2 : \tilde{X}_2 \longrightarrow X$$

basepoint $\tilde{x}_1$ basepoint $\tilde{x}_2$

are isomorphic as covering spaces $f : \tilde{X}_1 \to \tilde{X}_2$

taking $\tilde{x}_1 \mapsto \tilde{x}_2$

iff $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$.

Pf. $\boxed{\Longrightarrow}$ Clear by defn.

$\boxed{\Longleftarrow}$ Suppose

$$p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$$

$$\begin{tikzcd} & (\tilde{X}_2, \tilde{x}_2) \arrow[d, "p_2"] \\ (\tilde{X}_1, \tilde{x}_1) \arrow[ur, "\tilde{p}_1"] \arrow[r, "p_1"'] & (X, x_0) \end{tikzcd}$$

$\tilde{p}_1$ exists by lifting criterion.

$$\begin{tikzcd} & (\tilde{X}_1, \tilde{x}_1) \arrow[d, "p_1"] \\ (\tilde{X}_2, \tilde{x}_2) \arrow[ur, "\tilde{p}_2"] \arrow[r, "p_2"'] & (X, x_0) \end{tikzcd}$$

$\tilde{p}_2$ exists by lifting criterion.

By unique lifting property, $\tilde{p}_1 \tilde{p}_2 = \mathbb{1}$, $\tilde{p}_2 \tilde{p}_1 = \mathbb{1}$.
(these lifts fix basepoints.)

$\Longrightarrow$ $\tilde{p}_1$, $\tilde{p}_2$ are inverse covering space isoms.

$\blacksquare$