Our main thm: Mon

Thm 1.38 X PC, LPC, SLSC.

① $p:(\tilde{X},\tilde{x}_0)\to(X,x_0) \longmapsto p_*(\pi_1(\tilde{X},\tilde{x}_0))$

gives a bijection b/w

$$\left\{\begin{array}{l}\text{basepoint-preserving}\\ \text{isom classes of}\\ \text{path cntd covering spaces}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{subgroups of}\\ \pi_1(X,x_0)\end{array}\right\}$$

note both are posets!

covering $\tilde{\tilde{X}}\to\tilde{X}$ $\longleftrightarrow$ subgroup. $p_*(\pi_1(\tilde{\tilde{X}},\tilde{\tilde{x}}_0))\le p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

② If basepoints are ignored, this correspondence induces a bijection b/w

$$\left\{\begin{array}{l}\text{isom classes of}\\ \text{path cntd covering}\\ \text{spaces}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{conjugacy classes}\\ \text{of subgroups of}\\ \pi_1(X,x_0)\end{array}\right\}.$$

pf. of last statement

idea (this picture only for normal covers).

$$\begin{tikzcd}(\tilde{X},\tilde{x}_0) \arrow[rr, "f", "\cong"'] \arrow[dr, "p_0"'] & & (\tilde{X},\tilde{x}_1) \arrow[dl, "p_1"] \\ & (X,x_0) & \end{tikzcd}$$

f_* is basepoint change

Isom covers give conjugate subgroups.

Let $\tilde{\gamma}$ be a path $\tilde{x}_0 \rightsquigarrow \tilde{x}_1$.

$p \circ \tilde{\gamma} = \gamma$ some loop, let $g = [\gamma] \in \pi_1(X, x_0)$.

Let $H_i = p_{i*}(\tilde{X}, \tilde{x}_i)$ for $i = 0, 1$

- If $\tilde{f}$ is a loop at $\tilde{x}_0$, then $\overline{\tilde{\gamma}} \cdot \tilde{f} \cdot \tilde{\gamma}$ is a loop at $\tilde{x}_1$.

  $\Rightarrow g^{-1} H_0 g \subset H_1$

- Similarly, $g H_1 g^{-1} \subseteq H_0$

$\Rightarrow H_1 = g^{-1} H_0 g$.

Conjugate subgroups give isom covers

Suppose $H_1 = g^{-1} H_0 g$ and we have $\tilde{X} = X_{H_0}$ with basepoint $\tilde{x}_0$.

Choose γ representing g, lift to $\tilde{\gamma}$ w/ $\tilde{\gamma}(0) = \tilde{x}_0$ in $\tilde{X}$.

Let $\tilde{X}_1 = \tilde{X}$ and $\tilde{x}_1 = \tilde{\gamma}(1)$.

Note that $(\tilde{X}_1, \tilde{x}_1)$ w/ same covering map p satisfies $p_*(\tilde{X}_1, \tilde{x}_1) = H_1 = g^{-1} H_0 g$ indeed.

$\blacksquare$

Recall isomorphism of covering spaces

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f", "\cong"'] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

we can use f to identify the covers $\tilde{X}_1 \cong \tilde{X}_2$.

defn. For $p: \tilde{X} \to X$, the isoms $\tilde{X} \to \tilde{X}$ are called <u>deck transformations</u>.

<u>Observe</u>

① $G(\tilde{X}) = \{ \text{isoms } \tilde{X} \to \tilde{X} \}$ form a group

$\circ$, identity, inverse.

② by the <u>unique lifting property</u>,

a deck trans is entirely determined by where it sends a single point

(say, the basepoint $\tilde{x}_0$)

<u>eg.</u> $\mathbb{R} \to S^1$ $G(\tilde{X}) \cong \mathbb{Z}$ by vertical shift

- note takes $\tilde{x}_0$ to a different $\in p^{-1}(x_0)$

ie $\mathbb{Z} \subset \mathbb{R}$.

eg. $S^1 \to S^1$ n-sheeted cover

$z \mapsto z^n$

Then $G(\tilde{X}) = \{\text{rotations thru angles } 2\pi/n\} \cong \mathbb{Z}/n\mathbb{Z}$.

<u>dfn.</u> $p: \tilde{X} \to X$ is <u>normal</u> (aka "regular")

if for each $x \in X$ and $\tilde{x}, \tilde{x}' \in p^{-1}(x)$

$\exists$ a deck transformation taking $\tilde{x} \mapsto \tilde{x}'$.

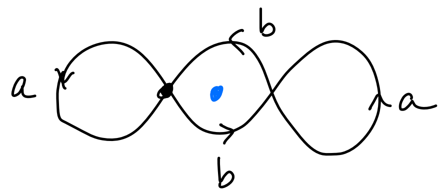
– maximal symmetry.

eg. $\mathbb{R} \to S^1$, $S^1 \to S^1$ are all normal.

eg. Consider our examples of covering space for $S^1 \vee S^1$

eg. (1), (2), (5)–(8), (11) are normal

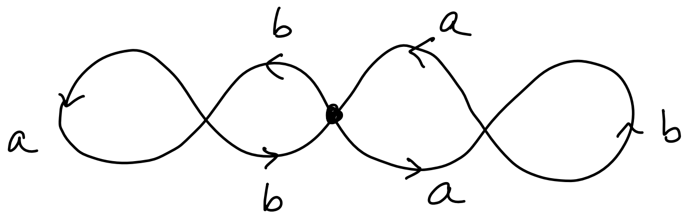
(7) $G = \mathbb{Z}/4\mathbb{Z}$ (8) $G = \mathbb{Z}/2 \times \mathbb{Z}/2$.

(1) 

$|p^{-1}(x_0)| = 2$ normal

$\Rightarrow G$ can only be $\cong \mathbb{Z}/2$

Indeed take rot about • by π as generator.

(3) 

not normal

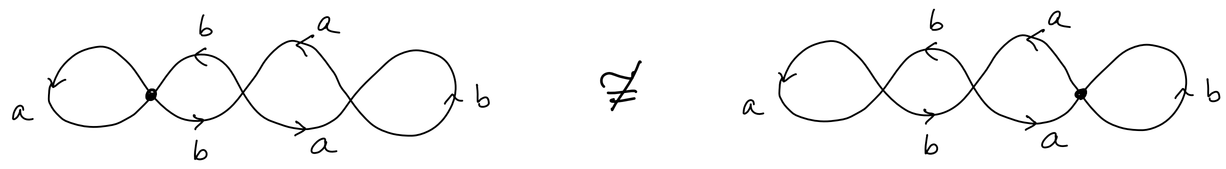
$|p^{-1}(x_0)| = 3$ so <u>at most</u> we could have 3 deck transformations.

But we cannot move $\tilde{x}_0$ to any other pt $\in p^{-1}(x_0)$

(no such homeo! if removed $\tilde{x}_0$ would get two disjoint pieces that look like $\propto$ ∞)

$G \cong 1$.

(4) Same with here, $G \cong 1$.



Recall before next time

<u>defn.</u> Let $S \subset G$. The <u>normalizer</u> of S in G is

$$N(S) = \{ g \in G \mid g S g^{-1} = S \}. \leq G.$$

ex $H \leq G$ is normal iff $N(H) = G$.

ex $H \leq N(H)$. (infact, $H \trianglelefteq N(H)$).

<u>Recall</u> w/ some clarification.

Given $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ $\boxed{p : \tilde{X} \to X}$ s.t. $p(\tilde{x}_0) = x_0$

- If $\tilde{X}$ normal then $\exists$ deck trans τ s.t.

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_0) \arrow[rr, "\tau"] \arrow[dr, "p"'] & & (\tilde{X}, \tilde{x}_1) \arrow[dl, "p"] \\
& (X, x_0) &
\end{tikzcd}$$

note the same "p"

$$\Rightarrow \quad \underbrace{p_*(\pi_1(\tilde{X}, \tilde{x}_0))}_{:= H_0} = \underbrace{p_*(\pi_1(\tilde{X}, \tilde{x}_1))}_{:= H_1} \subseteq \pi_1(X, x_0)$$

- If $\tilde{X}$ not normal, there might not be such τ.

  Let $[\gamma] \in \pi_1(X, x_0)$ s.t. $\tilde{\gamma}(0) = \tilde{x}_0$ and $\tilde{\gamma}(1) = \tilde{x}_1$.

  We showed then $H_1 = [\gamma]^{-1} H_0 [\gamma]$.

ex Could there be τ s.t. $\tilde{x}_0 \xrightarrow{\tau} \tilde{x}_1$ $(\tilde{x}_1 \neq \tilde{x}_0)$

but $p : \tilde{X} \to X$ not normal?

See examples @ end

Prop. 1.39

Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path-cnted covering space of a PC, LPC space X.

Let $H := p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$.

(a) p is normal iff $H \trianglelefteq \pi_1(X, x_0)$

$\exists$ deck trans taking any $\tilde{x} \rightsquigarrow \tilde{x}'$ where $\tilde{x}, \tilde{x}' \in p^{-1}(x)$

(b) $G(\tilde{X}) \cong N(H)/H$

group of deck trans

Rmks. Hence:

① $G(\tilde{X}) \cong \pi_1(X, x_0)/H$ if $\tilde{X}$ normal

② For the universal cover $\tilde{X}$, $G(\tilde{X}) \cong \pi_1(X)$.

Pf.

(a) $H_1 = [\gamma]^{-1} H [\gamma]$ so $[\gamma] \in N(H)$ iff $H_1 = H$. iff

By the lifting criterion, $\exists$ deck trans τ s.t. $\tilde{x}_0 \overset{\tau}{\mapsto} \tilde{x}_1$ (see prop 1.37)

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_1) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x}_0) \arrow[ur] \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

$p : \tilde{X} \to X$ is normal $\overset{?}{\Rightarrow}$ $\exists\, \tau$ $\tilde{x}_0 \rightsquigarrow$ any $\tilde{x}_1$

$\Leftrightarrow N(H) = \pi_1(X, x_0) \Leftrightarrow H \trianglelefteq \pi_1(X, x_0)$.

$\Leftarrow$:

$$\begin{tikzcd} \tilde{x}_1 \arrow[rr, bend left, "\tau' \circ \tau^{-1}"] & \tilde{x}_0 \arrow[l, "\tau"] \arrow[r, "\tau'"'] & \tilde{x} \end{tikzcd}$$

(b) Define $\varphi : N(H) \longrightarrow G(\tilde{X})$

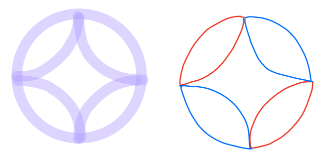
$[\gamma] \longmapsto$ the τ taking $\underset{\tilde{\gamma}(0)}{\tilde{x}_0} \rightsquigarrow \underset{\tilde{\gamma}(1)}{\tilde{x}_1}$

<u>check</u>: • φ is a homomorphism. ex

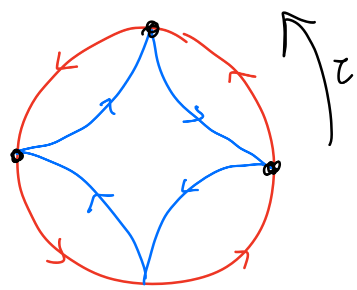
• φ is surjective (by pf. of (a))

$\ker\varphi = \{ [\gamma] \mid \underbrace{\tilde{\gamma}(0)}_{\tilde{x}_0} = \underbrace{\tilde{\gamma}(1)}_{\tilde{x}_1} \} = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H.$ ∎

<u>Examples</u> <u>normal</u>:



(7)

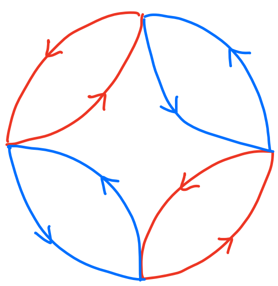


τ

(a) (b)

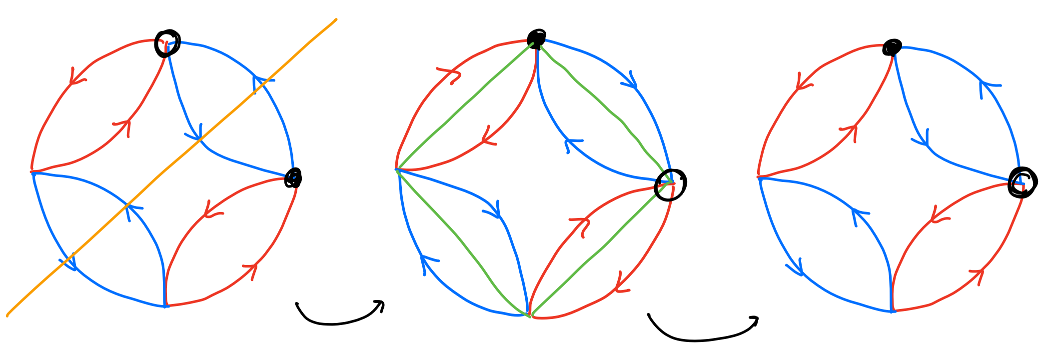
$G \cong \mathbb{Z}/4\mathbb{Z} = \langle \tau \mid \tau^4 = 1 \rangle$

(8)



σ = rotate by $180°$

τ = composition of following maps



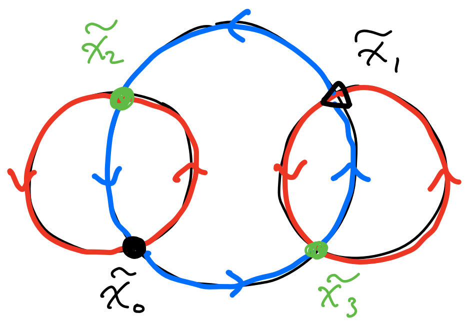
reflect across orange line

"reflect" across green square

Friday start here

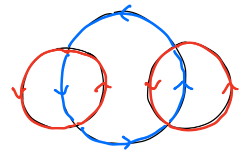
not normal:

(9)



$\exists\ \tilde{x}_0 \overset{\tau}{\rightsquigarrow} \tilde{x}_1$

but not to $\tilde{x}_2$ or $\tilde{x}_3$



$$p_*(\pi_1(\tilde{X},\tilde{x}_0)) = \langle a^2, b^4, ab, ba^2b^{-1}, bab^{-2} \rangle$$

these do not lift to loops.

HW 07: more ex as these.

Group Actions (Review from 150A)

defn Given a group G and a space Y, a (group) action of G on Y is a homomorphism

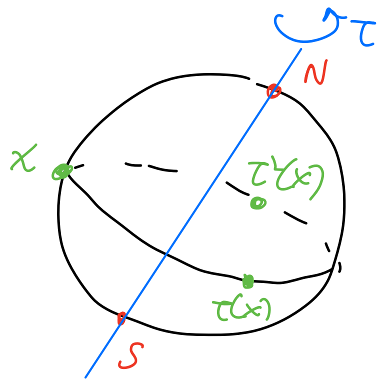
$$\rho : G \longrightarrow \mathrm{Homeo}(Y)$$

ie for each g, we have $\rho(g) : Y \xrightarrow{\cong} Y$

often written $g : Y \to Y$ for short.

e.g. $\mathbb{Z}/3\mathbb{Z} = \langle \tau \mid \tau^3 = 1 \rangle \curvearrowright S^2$ where

$\tau : S^2 \to S^2$ rotates by $2\pi/3$ about the axis through the N and S poles



- N, S are fixed points
- orbit of x is $\{x, \tau(x), \tau^2(x)\}$.

Hatcher's term; makes sense

defn. An action $G \curvearrowright Y$ is a covering space action if for each $y \in Y$ there is a nbhd $U \ni y$ s.t. $g_1(U) \cap g_2(U_2) \neq \emptyset$ iff $g_1 = g_2$. $(*)$

i.e. different sheets are all disjoint.

note: $\iff$ $g(U) \cap U \neq \emptyset$ iff $g = id$.

eg. For $p: \tilde{X} \to X$, $G(\tilde{X}) \curvearrowright \tilde{X}$ is a covering space action: the homeomorphic lifts $\tilde{U}_i$ of $\tilde{x}_i$ an evenly covered nbhd U of x are disjoint.

ex check details if unclear!

rmk Note that a covering space action is free.

$\longrightarrow$ If have cov space action of a manifold you will get a manifold back.

defn. Given $G \curvearrowright Y$,

- orbit of $y \in Y$: $Gy = \{g(y) \mid g \in G\}$
- $\underline{\underline{Y/G}}$ = <u>orbit space</u> of the action

notation

i.e. quotient by equiv relation $y \sim g(y)$

eg. For a normal covering space $\tilde{X} \to X$,

orbit space of $G(\tilde{X}) \curvearrowright \tilde{X}$ is $\cong X$.

since orbit of $\tilde{x}$ = <u>all</u> lifts of x.

<u>prop. 1.40</u> let $G \curvearrowright Y$ be a covering space action. (*)

(a) $p : Y \to Y/G$, $y \mapsto Gy$ is a normal covering space

(b) If Y is path-cntd, then G is the group of deck transformations.

(c) If Y is path-cntd & locally path cntd, then $G \cong \underbrace{\pi_1(Y/G) / p_*(\pi_1(Y))}_{\text{add basepoints appropriately.}}$

Pf. (Sketch)

"$\tilde{U}$" in old notation.

(a) <u>covering space</u>: indeed the U as in (*) map to an evenly covered nbhd $p(U)$.

<u>normal</u> by construction $\exists\, g$ taking each U to $g(U)$

(b) Clear that $G \subset$ group of deck trans. "Deck"

If $f \in$ Deck, and $y \overset{\tilde{\gamma}}{\rightsquigarrow} f(y)$ by a path (since Y path cntd)

$\Longrightarrow \exists\, g$ (equiv class of $[\gamma] = [p \circ \tilde{\gamma}]$)

s.t. $g(y) = f(y)$

But Y path cntd so since they agree on one pt they are the same deck trans: $g = f$.

(c) b/c $Y \to Y/G$ is normal; use prop. 1.39 (b).

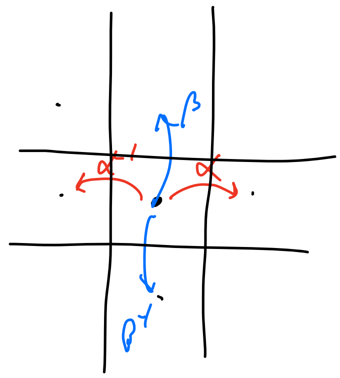
$\boxtimes$

Rmk When Y is simply cntd, locally path cntd (i.e. a universal cover for any Y/G where $G \curvearrowright Y$ is a cov. space action)

then $\pi_1(Y/G) \cong G$ by construction.

EXAMPLES

① $\mathbb{R}^2$



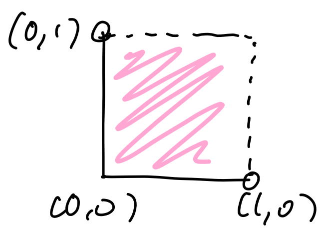
$\alpha(x,y) = (x+1, y)$
$\beta(x,y) = (x, y+1)$ } translations

for any pt take $U \ni x$ to be ball of radius < 1.

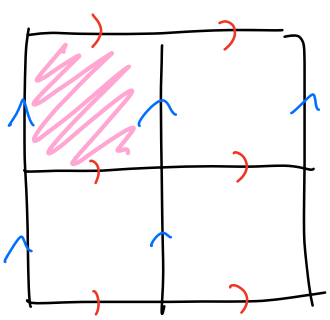
Then clear that (*) holds.

$G = \mathbb{Z} \times \mathbb{Z}$ gen by α, β.

$\mathbb{R}/G \cong T^2$ $\qquad \pi_1(T^2) \cong \mathbb{Z}\times\mathbb{Z}$ indeed.



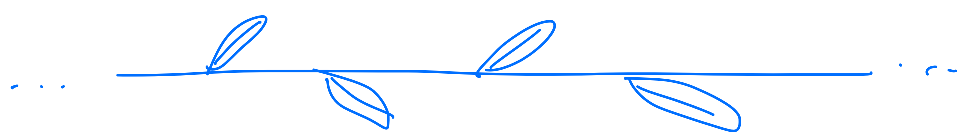
is a fundamental domain
contains a rep of each orbit.

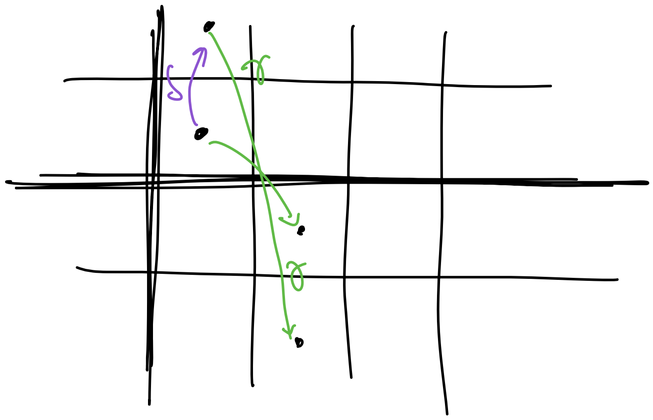


Use this to see we get a torus

② Slight modification:

glide reflection: a symmetry of

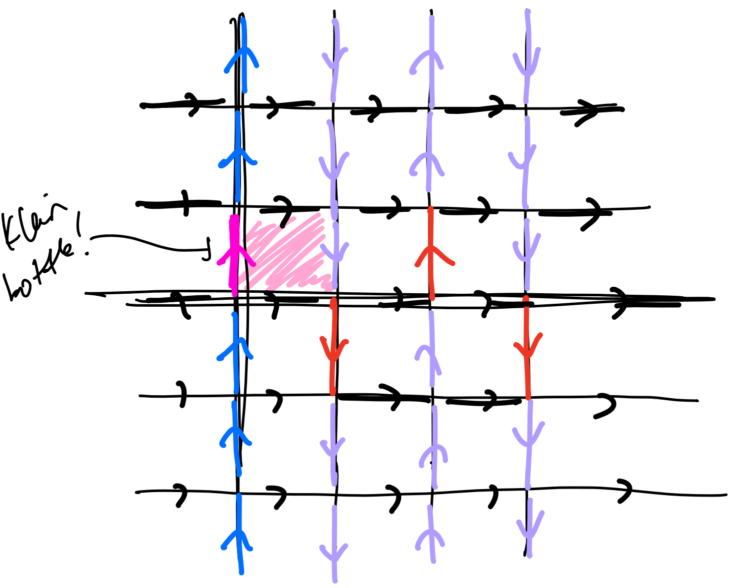




$\gamma(x,y) = (x+1, -y)$

$\delta(x,y) = (x, y+1)$

Let's see how sides of squares are identified.



- γ generates subgroup $\cong \mathbb{Z}$
- δ ______

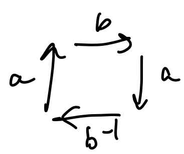
- $\gamma^2(x,y) = (x+2, y)$ and $\langle \gamma^2 \rangle \cong \mathbb{Z}$ in G.
- $\langle \gamma^2, \delta \rangle \subset G$ is index 2, $\cong \mathbb{Z} \times \mathbb{Z} \cong \pi_1(T^2)$.

$\Longrightarrow$ see there is a 2-sheeted cover of Klein bottle by T^2.

- colored edges id'd.
- black edges id'd.

- On one hand

$$\pi_1(\text{Klein}) = \langle a, b \mid abab^{-1} \rangle$$



- on the other hand also see that

γ, then β, then γ^{-1} then β = id deck trans

ie $\beta\gamma^{-1}\beta\gamma = 1$.