π_k groups

Whitehead theorem

CW cpxs, approx

Fiber bundles (Serre spec. sq)

Mon

π_k basics

Let $I = [0,1]$ as usual. I^n = n-dim'l cube.

<u>defn</u> <u>nth homotopy group</u> of (X, x_0)

$\pi_n(X, x_0) = \{$ hpy classes of maps

$f: (I^n, \partial I^n) \longrightarrow (X, x_0)$

$\equiv f: (S^n, s_0) \longrightarrow (X, x_0)$

where htpies satisfy $f_t(\partial I^n) = x_0 \ \forall t \}$.

eg. • π_1; agree?

• π_0: $I^0 = \bullet$, $\partial I^0 = \emptyset$. By convention.

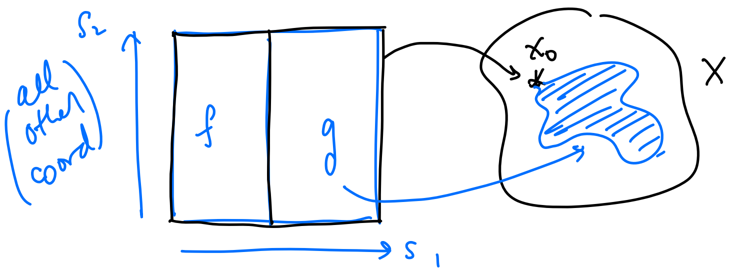
$\Rightarrow \pi_0(X, x_0)$ = set of <u>path</u> components of X.

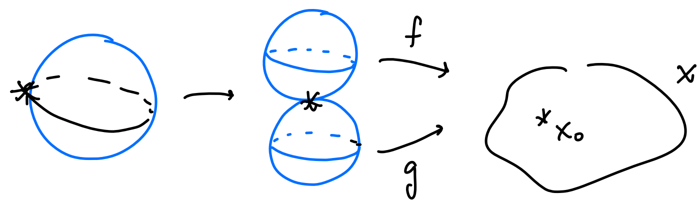
htpy of pt maps.

Group structure of π_n, $n \geq 2$.

- $$(f+g)(s_1, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & s \in [0, 1/2] \\ g(2s_1 - 1, s_2, \dots, s_n) & s \in [1/2, 1] \end{cases}$$

Idea $n=2$:



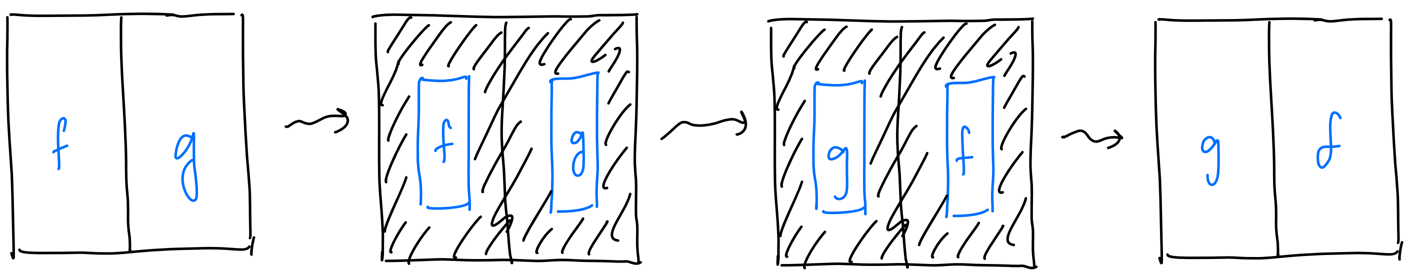


* check well defined ?

- inverse: $f(s_1, \dots, s_n) = f(1-s_1, s_2, \dots, s_n)$

- Abelian (hence additive notation)

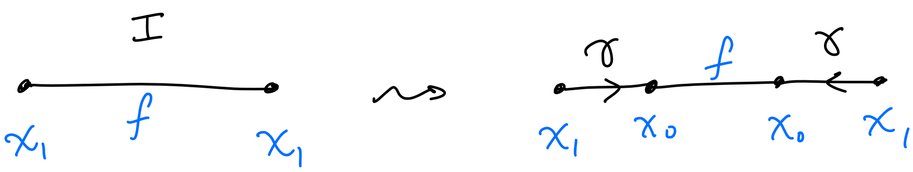
Black: sent to basepoint



* no room when $n=1$; yes room when $n \geq 2$

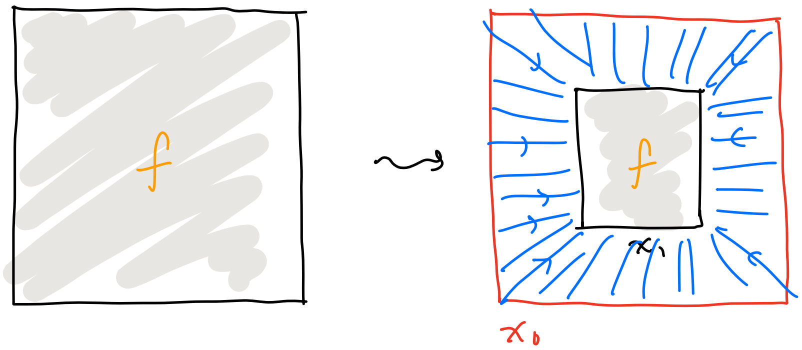
Change of basepoint

e.g. π_1:



"$\gamma \circ f \circ \gamma^{-1}$"

Same idea: π_k



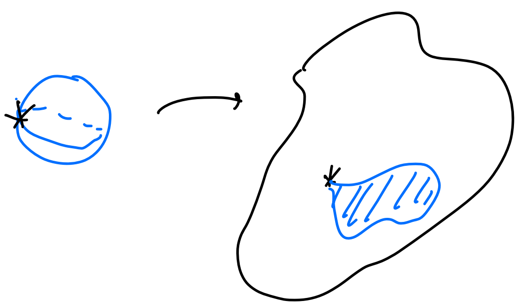
Write as "γf"

⚠ Diff. notation from π_1

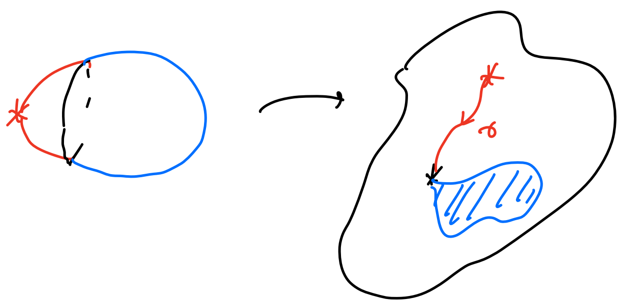
black $\to x_1$

all blue arcs $\to \gamma$.

Cartoon ($n=2$)

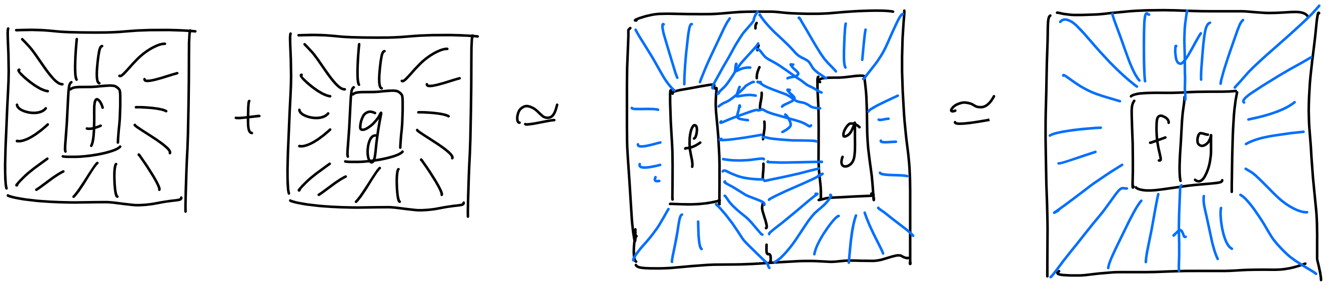


becomes



Properties:

① $\gamma(f+g) \simeq \gamma f + \gamma g$



② $(\gamma\eta) f \simeq \gamma(\eta f)$ ✓

③ $1 f \simeq f$

$\Rightarrow$ Define change-of-basepoint hom $\beta_\gamma([f]) = [\gamma f]$.

- $\beta_{\bar\gamma}$ is β_γ^{-1}
- So if X is path cntd, $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ and we can write $\pi_n(X)$

π_1 action

- Also note if $\gamma \simeq \gamma'$ (path htpy) then $\beta_\gamma = \beta_{\gamma'}$.
- For loops γ based at x_0, we have an assignment $\underset{\pi_1(X, x_0)}{[\gamma]} \mapsto \underset{\mathrm{Aut}(\pi_n(X, x_0))}{\beta_\gamma}$

  with $\beta_{\gamma\eta} = \beta_\gamma \circ \beta_\eta$ (think through!)

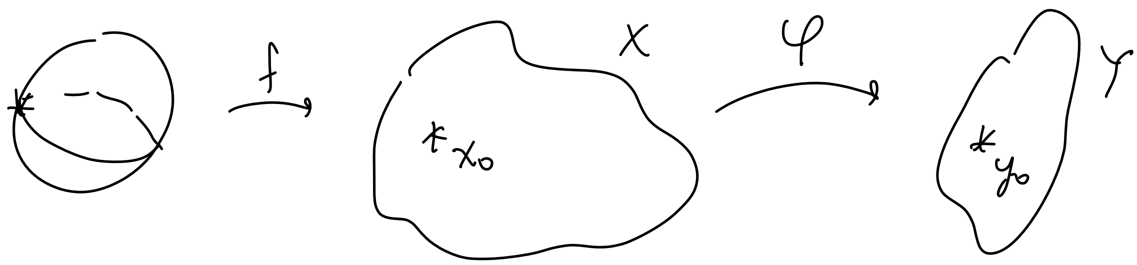
$\Rightarrow$ $\pi_n(X, x_0)$ is a $\mathbb{Z}[\pi_1]$-module!

Wed

Functoriality

$\varphi : (X, x_0) \longrightarrow (Y, y_0)$ induces

$$\varphi_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0)$$
$$[f] \longmapsto [\varphi \circ f]$$



Check/Observe

- φ_* is well defined $f \simeq g \Rightarrow \varphi f \simeq \varphi g$.
- φ_* is a homomorphism

- $(\varphi\psi)_* = \varphi_* \psi_*$, $\mathbb{1}_* = \mathbb{1}$.
- if φ_t is a htpy $\varphi_0 \rightsquigarrow \varphi_1 : (X, x_0) \to (Y, y_0)$ then $\varphi_{0*} = \varphi_{1*}$.

Behavior in rel to Covering spaces

prop 4.1 A covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces isoms

$$p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$$

for $n \geq 2$.

Pf.

p_* is surjective:

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (S^n, s_0) \arrow[r, "f"'] \arrow[ur, "\exists \tilde{f}"] & (X, x_0) \end{tikzcd}$$

by lifting criterion, since $\pi_1(S^n) = 1$.

Then $p_*([\tilde{f}]) = [f]$.

p_* is injective

(same pf as before)

(Recall when we showed $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective too.)

If $p_*([\tilde{f}]) = [c]$ $\quad (f = p \circ \tilde{f})$

then $\exists$ htpy $f \simeq c$.

By htpy lifting property (prop 1.30) there exists (a unique) htpy showing $\tilde{f} \simeq \tilde{c}$ (lift of const is const).

Rmk. Suppose (X, x_0) has a contractible universal cover $(\tilde{X}, \tilde{x}_0)$ then

$$\pi_n(X, x_0) = 0 \quad \forall\, n \geq 2. \Big\} \text{ "aspherical"}$$

eg. Univ. cover of $T^k = \underbrace{S^1 \times \cdots \times S^1}_{n}$ is $\mathbb{R}^k$

Since $T^k = \mathbb{R}^k / (\text{action of})\ \mathbb{Z}^k$.

$$\Rightarrow \pi_n(T^k) = 0 \quad \forall\, n \geq 2.$$

eg. On the other hand,

<u>Fact</u> $\pi_2(S^2) \cong \mathbb{Z}$. $\quad (\pi_k(S^k) \cong \mathbb{Z})$.

$\Rightarrow \pi_2(\mathbb{R}P^2) \cong \mathbb{Z}$ as well.

Finally, small fact, last easy thing about π_n:

Prop 4.2 For $\prod_\alpha X_\alpha$ where each X_α is path-conn'ted, $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$ $\forall n$.

indeed π_1 as well

Pf $f: Y \to \prod_\alpha X_\alpha$ is the same data as $\{f_\alpha : Y \to X_\alpha\}$.

$$\Phi: \pi_n\left(\prod_\alpha X_\alpha, (x_\alpha)\right) \longrightarrow \prod_\alpha \pi_n(X_\alpha, x_\alpha)$$

- $f: (S^n, s_0) \to \left(\prod_\alpha X_\alpha, (x_\alpha)\right)$

  $\longleftrightarrow \{f_\alpha : (S^n, s_0) \to (X_\alpha, x_\alpha)\}$

- If $F: S^n \times I \to \prod_\alpha X_\alpha$ is a htpy

  where $F(\{s_0\} \times I) = (x_\alpha)$.

  this is the same data as

  $$\{F_\alpha : S^n \times I \to X_\alpha\}$$

  where $F_\alpha(\{s_0\} \times I) = x_\alpha$ $\forall \alpha$.

$\boxtimes$

defn (X, x_0) is <u>n-connected</u> if
$\pi_i(X, x_0) = 0$ for all $i \leq n$.

- "0-connected" = path-connected eg. S^1
- "1-connected" = simply-connected eg. S^2
- S^n is $(n-1)$-connected.

Just like in a previous thm, we can remove mention of a basepoint when discussing connectedness:

<u>prop.</u> TFAE (characterizing n-connectedness)

1. Every map $S^i \to X$ is htpc to a const map.
2. Every map $S^i \to X$ extends to a map $D^{i+1} \to X$.
3. $\pi_i(X, x_0) = 0 \quad \forall\, x_0 \in X$

Similar proof to π_1 case.