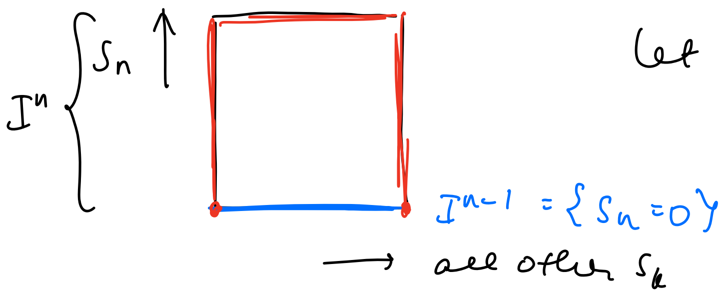
Mon 12/1

Relative Homotopy Groups (maybe put use next page)

$\pi_n(X, A, x_0)$ for a pair (X, A) with $x_0 \in A$.



Let $J^{n-1} = \text{closure}(\partial I^n - I^{n-1})$

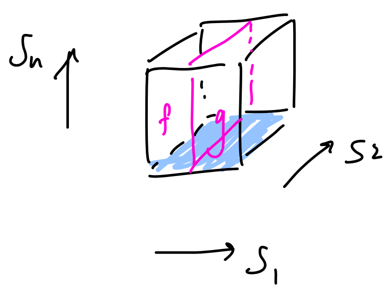
- For $n \geq 1$: $\pi_n(X, A, x_0) = \{$ htpy classes of maps

$$(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$$

interior of I^{n-1} allowed to go into A
but J^{n-1} must be sent to basepoint.

through htpies of the same form $\}$.

- Same sum operation, except you can't use s_n for the operation.



So $\pi_n(X, A, x_0)$ is a gp for $n \geq 2$
& abelian for $n \geq 3$.

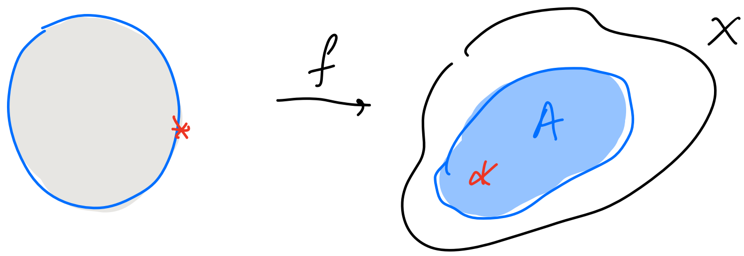
- $\pi_n(X, x_0) \cong \pi_n(X, x_0, x_0)$ canonically.

identified with

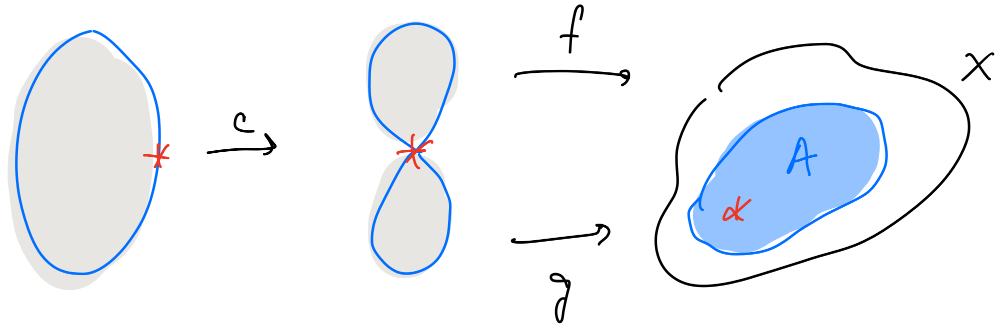
Equivalently :

$\pi_n(X, A, x_0)$ = htpy classes of maps

$$(D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$$



Addition :



Compression criterion

Prop. A map $f: (D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$ is trivial in $\pi_n(X, A, x_0)$ iff it is htpc rel S^{n-1} to a map with image contained in A.

(Believable; pf easy, omitted).

Functoriality π_n is functorial wrt maps

$\varphi : (X, A, x_0) \to (Y, B, y_0)$

- These φ_* are homomorphisms for $n \geq 2$
- Same usual properties, and $\varphi_* = \psi_*$ if $\varphi \simeq \psi$ through maps $(X, A, x_0) \to (Y, B, y_0)$

Algebra Aside Exact Sequences.

* note: will use "0" a trivial group out of habit (usually modules)

Work with groups.

defn A sequence of homomorphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

is <u>exact</u> if $\forall n$, $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$.

A sequence

$$0 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 0$$ ← trivial group

is called a SES.

eg. ① $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$

This is a "<u>short exact sequence</u>".

② $\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{id} \mathbb{Z} \longrightarrow 0$

③ $0 \longrightarrow K \xrightarrow{i} G \xrightarrow{\varphi} H \longrightarrow 0$

where $K = \ker \varphi$

This is a useful way to characterize maps

- $0 \to A \xrightarrow{\alpha} B$

  is exact iff

  α is injective

- $A \xrightarrow{\alpha} B \to 0$

  is exact iff

  α is surjective.

Habit: $0 \to A \to B$
b/c usually
thinking about modules
over a ring + the
group structure is
abelian

Long Exact Sequence

Why define these? To compute π_n's for spaces using known π_n's of spaces we already know.

$i : (A, x_0) \hookrightarrow (X, x_0)$

$j : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$

thm 4.3 There is a long exact sequence

$$\to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \pi_{n-1}(X, x_0) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_0(X, x_0).$$

where the boundary map ∂ comes from restricting maps

$$(D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0) \text{ to } S^{n-1}.$$

- ker = im still makes sense when at the end of the sequence we have sets rather than groups.

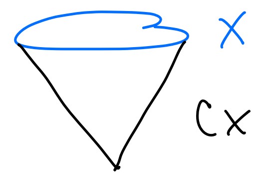
(see Whitehead's theorem for better use)

* not an eg of how we use to compute $\pi_n(Y)$

but this is how LES are sometimes used.

eg. Recall the cone $CX = X \times I / X \times \{0\}$.

View $X \subset CX$ as $X \times \{1\}$.



Since $CX \simeq *$, $\pi_n(CX) = 0$

So the LES tells us for $n \geq 1$,

$$\pi_n(CX, X, x_0) \cong \pi_{n-1}(X, x_0)$$

$\Rightarrow$ we can realize any group G as a relative π_2 by choosing X s.t. $\pi_1(X) \cong G$.

Recall long exact sequence: ker = im at each group.

$$\to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0)$$

$$\xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \pi_{n-1}(X, x_0) \to \cdots$$

$$\cdots \to \pi_0(X, x_0).$$

Use this on board during pf

Use this to prove a lemma we'll need.

Lemma A: If $f : X \hookrightarrow Y$ is an inclusion and f_* is an isom, then $\pi_n(Y, X, x_0) = 0$.

$(\forall\, x_0 \in X)$

Pf. use LES of htpy groups for the pair (Y, X)

* Student 1st time seeing this calc → slowly

$$
\begin{tikzcd}
\pi_n(X,x_0) \arrow[r, "f_*", "\cong \text{①}"'] & \pi_n(Y,x_0) \arrow[r, "j_*"] & \pi_n(Y,X,x_0) \text{③} \arrow[dll, "\partial"'] \\
\pi_{n-1}(X,x_0) \arrow[r, "f_*", "\cong \text{②}"'] & \pi_{n-1}(Y,x_0) \arrow[r, "j_*"] & \pi_{n-1}(Y,X,x_0)
\end{tikzcd}
$$

① f_* is $\cong$

$\Rightarrow \operatorname{im} f_* = \pi_n(Y, x_0)$

$\Rightarrow \ker j_* = \pi_n(Y, x_0)$

$\Rightarrow \operatorname{im} j_* = 0$

② f_* is $\cong$

$\Rightarrow \ker f_* = 0$

$\Rightarrow \operatorname{im} \partial = 0$

$\Rightarrow \ker \partial = \pi_n(Y, X, x_0)$

③ But $\ker \partial = \operatorname{im} j_* \Rightarrow \pi_n(Y, X, x_0) = 0$.

∎

Write small on side of board

<u>thm. 4.5</u> (Whitehead's theorem)

(a) If a map $f : X \longrightarrow Y$ between connected CW cpxs induces isoms

$$f_* : \pi_n(X) \longrightarrow \pi_n(Y) \quad \text{for all } n,$$

then f is a homotopy equivalence.

(b) If f is (additionally) the inclusion of a subcomplex $X \hookrightarrow Y$, then X is a deformation retract of Y.

The proof depends on a useful Lemma:

Lemma B $^{(4.6)}$ (Compression Lemma)

Let (X,A) be a CW pair and (Y,B) any pair w/ $B \neq \emptyset$.

For each n such that $(X-A)$ has cells of dim n,

Suppose $\pi_n(Y, B, y_0) = 0 \quad \forall y_0 \in B$.

Then every map $f:(X,A) \longrightarrow (Y,B)$ is htpc rel A

to a map $X \to B$.

When $n=0$, "$\pi_n(Y,B,y_0)=0$" means

(Y,B) is "0-connected", ie each path cpnt

of Y contains points in B.

(Pf. by induction, omitted)

Lemma B reworded (X,A) CW, $(Y, B \neq \emptyset)$

If $\forall n$ where $X-A$ has n-cells

we have $\pi_n(Y,B,y_0)=0 \quad \forall y_0 \in B$,

then

$f:(X,A) \longrightarrow (Y,B)$ is htpc rel A

to a map $f': X \to B$.

squish into B,

keeping A always mapping into B.

small on board w/ them?

Pf of Whitehead's theorem

Special case (b):

By lemma A, $\pi_n(Y, X, x_0) = 0 \quad \forall x_0 \in X$.

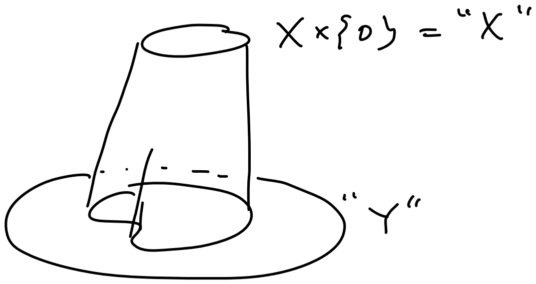
By lemma B applied to $id : (Y, X) \to (Y, X)$,

id_Y is htpc rel X to a map $Y \to X$,

ie. there is a deformation retraction of Y onto X.

General case (a): Use the mapping cylinder:

For $f : X \to Y$, $\quad M_f = X \times I \sqcup Y / (x,1) \sim f(x)$



$i_X : X \hookrightarrow M_f$

$i_Y : Y \hookrightarrow M_f$

- $\exists$ deformation retraction $r_t : M_f \to M_f$ onto Y:

  $r_0 = id_{M_f}$, $\quad r_t|_Y = i_Y$, $\quad img\, r_1 = Y$.

- let $r_Y : M_f \to Y$ be r_1 with restricted target

  Then $f = r_Y \circ i_X$:

$$\begin{tikzcd} X \arrow[r, hook, "i_X"] \arrow[rr, bend right, "f"'] & M_f \arrow[r, "r_Y"] & Y \end{tikzcd}$$

Strategy

- Since r_Y is a htpy equivalence, to show f is a htpy equiv, ISTS (it suffices to show) i_X is a htpy equiv.
- To show i_X is a htpy equiv.,

$$\boxed{\text{ISTS there is a deformation retraction of } M_f \text{ onto } X}$$

- We have the assumption that

  f induces isoms f_* on all htpy groups

Claim By the LES on π_n, this is equivalent to the assumption

$$\pi_n(M_f, X) = 0 \quad \forall n \geq 1$$

Pf. Claim (idea)

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[r, hook, "i_{X*}"] \arrow[ddr, "f_*"', "\cong"] & \pi_n(M_f, x_0) \arrow[r, "j_*"] \arrow[d, shift right, "\cong"'] & \pi_n(M_f, X, x_0) \\
& \pi_n(M_f, f(x_0)) \arrow[u, shift right, "\text{basepoint change}"'] \arrow[d, shift right, "r_*"'] & \\
& \pi_n(Y, f(x_0)) \arrow[u, shift right, "i_{Y*}"'] \arrow[uur, "j'_*"'] &
\end{tikzcd}$$

- Note we also have "π_0"$(M_f, X) = 0$ since M_f is path-cntd.

Aside: If f is cellular, ie f takes X^n to Y^n for all n, then (M_f, X) is a CW pair, and we are back to the special case (b)

M_f deformation retracts to X, so by the strategy we are done.

If f is not cellular, we could use cellular approximation (see Chp 4).

Or we could use the argument below.

We will use Lemma B twice to build two htpies relating id_{M_f} to a retraction $r_X : M_f \to X$.

Let ♥ = the assumption that $\pi_n(M_f, X) = 0 \quad \forall n$.
and $X \neq \emptyset$.

① By Lemma B, $\exists$ htpy $\{g_t\}$ rel X from

$$(X \cup Y, X) \xhookrightarrow{i} (M_f, X)$$

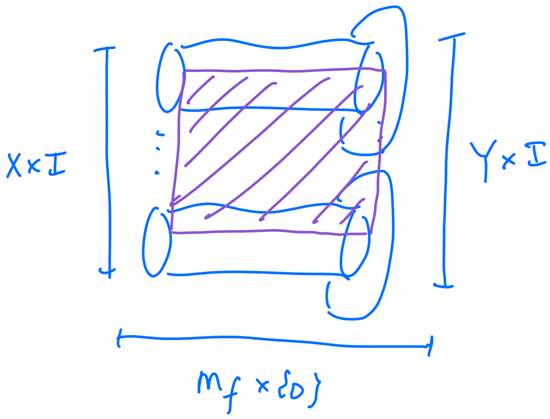
to a map $X \cup Y \to X$.

lemma applies: $(X \cup Y, X)$ is CW pair, ♥

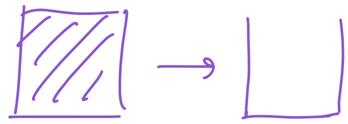
- **Claim** $(M_f, X\cup Y)$ satisfies the HEP

idea: there is a retraction

$$M_f \times I \longrightarrow M_f \times \{0\} \cup (X\cup Y)\times I$$



Each $x\in X$ defines a square $\cong I\times I$ that has a retraction



Together, these define the retraction we want. //

- HEP $\Rightarrow$

$$\begin{tikzcd}
 & & M_f \\
(X\cup Y)\times I \arrow[r] \arrow[urr, bend left, "g_t"] & M_f\times I \arrow[ur, "\tilde g_t"] & \\
(X\cup Y)\times\{0\} \arrow[u, hook] \arrow[r, hook] & M_f\times\{0\} \arrow[u, hook] \arrow[uur, bend right, "\tilde g_0 = id_{M_f}"'] &
\end{tikzcd}$$

where $g_0 = i : (X\cup Y, X) \longrightarrow (M_f, X)$

$\simeq$ rel X $\quad g_1 : X\cup Y \to M_f$ w/ $\operatorname{im} g_1 = X$

and $\{g_t\}$ is a htpy of maps $M_f \to M_f$

where $\tilde g_0 = id_{M_f}$

$\tilde g_1$ w/ $\tilde g_1(X\cup Y) = g_1(X\cup Y) = X$ $\quad \tilde g_1|_X = id_X$

$\tilde g_t|_X = id_X \quad \forall t.$

Hence: $id_{M_f} \simeq \tilde g_1$ where $\tilde g_1|_X = id_X$, $\tilde g_1(X\cup Y) = X$.

② Let φ be the composition

$$\underbrace{(X\times I \sqcup Y,\ X\times \partial I \sqcup Y)}_{\text{a CW pair!}} \xrightarrow[\text{gluing map}]{q} (M_f, X\cup Y) \xrightarrow{\tilde{g}_1} (M_f, X)$$

Lemma 8 applies to φ:

$(X\times I \sqcup Y,\ X\times\partial I \sqcup Y)$ is a CW pair,

$\Rightarrow$ $\exists$ a htpy $\{\varphi_t\}$ rel $X\times\partial I\sqcup Y$ to a map

$X\times I\sqcup Y \longrightarrow X$

$$\{\varphi_t\} : (X\times I\sqcup Y,\ X\times\partial I\sqcup Y) \longrightarrow (M_f, X)$$

where $\varphi_0 = \varphi$

$(\star)$ $\varphi_t|_{X\times\partial I\sqcup Y} = \varphi|_{X\times\partial I\sqcup Y}$

$\varphi_1(X\times I\sqcup Y) = X$

- By $(\star)$, the htpy φ_t factors through M_f in the following sense: (we can always glue $X\times\{1\}$ to Y)

$$\begin{tikzcd}
(X\times I\sqcup Y,\ X\times\partial I\sqcup Y)\times I \arrow[d, "q\times id"] \arrow[dr, "\varphi_t"] & \\
(M_f, X\cup Y)\times I \arrow[r, "\overline{\varphi}_t"'] & (M_f, X)
\end{tikzcd}$$

where $\overline{\varphi}_0 = \tilde{g}_1$ (since $\varphi_0 = \varphi = \tilde{g}_1\circ q$),

$\overline{\varphi}_1$ is a retraction onto X (recall $\tilde{g}_1|_X = id_X$)

$\Rightarrow$ $\tilde{g}_0 = id_{M_f} \simeq \tilde{g}_1 \simeq \overline{\varphi}_1$, a retraction to X $\blacksquare$