MAT 215A Fall 2025

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Exam 1 Solutions

Comments from the instructor are written in italics.

1. Let X be a topological space and let $x, y \in X$. Define $e_x : * \to X$ be the function that sends the single point in $*$ to $x \in X$; similarly define $e_y : * \to X$ to be the map that sends the point to y.

   Prove that x and y are connected by a path if and only if $e_x \simeq e_y$.

   Solution. ($\Longrightarrow$) Suppose there is a path $f(s) : I \to X$ where $f(0) = x$, $f(1) = y$. Define a homotopy $g_t : * \to X$ by $g_t(*) = f(t)$. Since $*$ is a 1-point space, each g_t is continuous in s. This homotopy is continuous in t because $f(t)$ is continuous. We have $g_0 = e_x$ and $g_1 = e_y$, so $e_x \simeq e_y$.

   ($\Longleftarrow$) If $g_t : * \to X$ is a homotopy with $g_0 = e_x$ and $g_1 = e_y$, then $f(s) = g(s)$ gives a continuous path from $f(0) = x$ to $f(1) = y$ in X.

2. (a) Write down the definition of "deformation retraction."

   Solution. Let (X, A) be a pair of spaces. We say a homotopy $f_t : X \to X$ is a *deformation retraction* of X onto A if $f_0 = id_X$, $f_1(X) = A$, and $f_t(a) = a$ for all $a \in A, t \in I$.

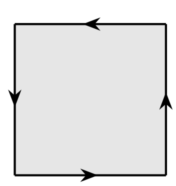
   (b) Construct an explicit deformation retraction from $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

   Solution. Define a straight-line homotopy taking $x \in \mathbb{R}^n \backslash \{0\}$ to $\frac{x}{\|x\|} \in S^{n-1}$ by

   $$f_t(x) = (1-t)x + t\frac{x}{\|x\|}.$$

   Since $\|x\| \neq 0$, this is continuous in x. This is linear and hence continuous in t. Also observe that the straight-line path of each point avoids the missing origin in $\mathbb{R}^n$.

3. Let X be the space obtained by identifying the four sides of a square $[0, 1]^2$ as shown:



   *Observe that X is **not** homeomorphic to the torus or Klein bottle; to see this, consider neighborhoods of a point in the image of the boundary of the square.*

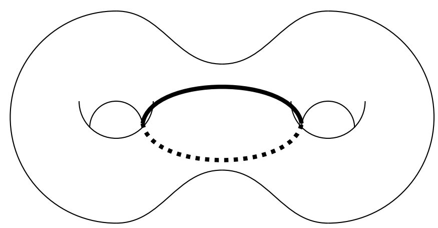
   Write down a CW structure for X, describing the attaching maps explicitly.

Solution. Let $X^0 = \{v\}$, and attach a single 1-cell e so that $X^1 \cong S^1$. View e as a path $e : I \to X^1$. To obtain X^2, attach a single 2-cell F, identified with the filled square $I \times I$ via a homemorphism $D^2 \to I \times I$, using the piecewise attaching map $\phi_f : \partial F \to X^1$ where

$$\begin{aligned} \phi_F(x,0) &= e(x) \\ \phi_F(1,y) &= e(y) \\ \phi_F(x,1) &= e(1-x) \\ \phi_F(0,y) &= e(1-x). \end{aligned}$$

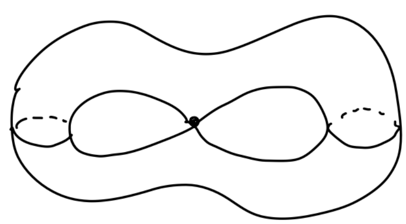
Thus ∂F wraps around a four times in the same direction.

4. Let X be the genus-2 surface shown below, and let A be the thick circular subspace shown. Prove that the quotient space X/A is homotopy equivalent to $S^1 \vee T^2$.

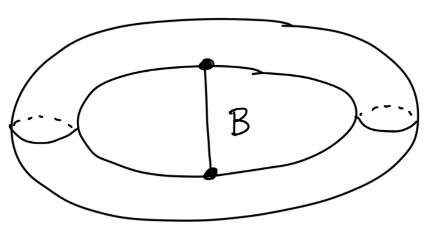


Recall that T^2 is the torus. You may mainly draw pictures in this proof. However, you must justify why each step is a homotopy equivalence.

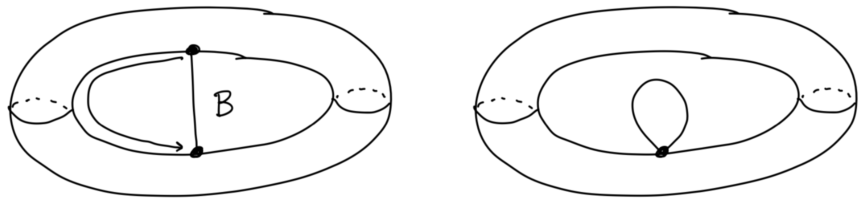
Solution. The quotient space X/A is obtained by crushing A to a point:



This is homotopy equivalent to the following space Y, because B is a contractible subspace of Y, and $X/A \cong Y/B$:



We can build Y by attaching an arc B to a torus using a different but homotopic attaching map:



This space is a wedge of a circle and a torus $S^1 \vee T^2$.