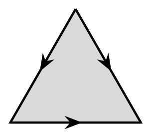
MAT 215A Fall 2025
Instructor: Melissa Zhang
Exam 2 Solutions

1. Let X be the space obtained by identifying the three sides of an equilateral triangle as shown:



Compute $\pi_1(X)$. *Be careful. No points will be given for computing π_1 of the wrong space.*

Solution. The CW-complex X is built by attaching a single 1-cell to a single 0-cell x_0 to form $X^1 \cong S^1$. Let γ be a loop based at x_0 that travels along the 1-cell once. We form $X = X^2$ by attaching a single 2-cell f to X^1 along $\gamma \cdot \gamma \cdot \overline{\gamma} \simeq \gamma$. By the Seifert-van Kampen theorem, the fundamental group has the presentation $\pi_1(X, v) = \langle \gamma \mid \gamma \rangle$, which is the trivial group.

2. For a covering space $p : \tilde{X} \to X$ and a subspace $A \subset X$, let $\tilde{A} := p^{-1}(A)$. Show that the restriction $p|_{\tilde{A}} : \tilde{A} \to A$ is a covering space.

Solution. To check that $p|_{\tilde{A}}$ is a covering map, we need only check the covering space condition: that every point $a \in A$ has an evenly covered open neighborhood V_a.

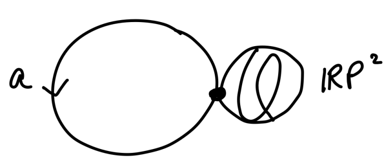
Since p is a covering map, a has an evenly covered neighborhood U_a. We claim that $V_a = U_a \cap \tilde{A}$ is evenly covered under $p|_{\tilde{A}}$. Indeed,

- any two sheets V, V' of $p^{-1}(V_a)$ are disjoint because they are contained in disjoint sheets U, U' of $p^{-1}(U_a)$;
- the restriction of p to each sheet V of $p^{-1}(V_a)$ is a homeomorphism because it is a restriction of the homeomorphism $p|_U$ where $V \subset U$, a sheet of $p^{-1}(U_a)$.

3. Find two non-homeomorphic 2-sheeted, path-connected covering spaces of $X = S^1 \vee \mathbb{R}P^2$. Prove that the covers are indeed not homeomorphic.

You do not need to explicitly prove your covers are 2-sheeted; just make sure to clearly indicate how your covering map sends pieces of $\tilde{X}$ to pieces of X, as we did in class when we studied covers of $S^1 \vee S^1$.

Here is a cartoon of X:

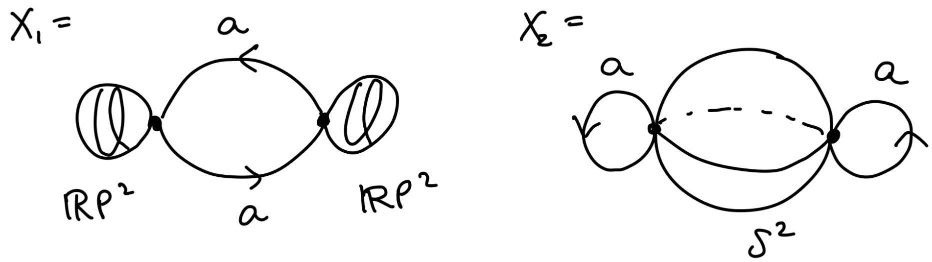


Here is one possible solution.

Solution.

Let x_0 be the wedge point. Let a be a loop whose homotopy class generates $\pi_1(S^1, x_0) \cong \mathbb{Z}$ and let b be a loop that generates $\pi_1(\mathbb{R}P^2, x_0) \cong \mathbb{Z}/2\mathbb{Z}$. We may construct 2-sheeted covers by unwrapping these loops by a factor of 2.

Consider the following two covering spaces of $\tilde{X}_1$ and $\tilde{X}_2$ of X:



The first is obtained by unwrapping the curve a by a factor of 2, and the second is obtained by taking the universal cover of $\mathbb{R}P^2$ and attaching two copies of a, one at each lift of $\tilde{x}_0$.

To show that these covers are not homeomorphic, we can compute their fundamental groups. Since $\tilde{X}_1$ is homotopy equivalent to $S^1 \vee \mathbb{R}P^2 \vee \mathbb{R}P^2$, we have $\pi_1(X_1) \cong \mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. On the other hand, $\tilde{X}_2 \simeq S^1 * S^1 * S^2$, which has fundamental group $\pi_1(X_2) \cong \mathbb{Z} * \mathbb{Z}$.

Since $\pi_1(X_1)$ has elements of order 2 but $\pi_1(X_2)$ does not, they are not isomorphic, and hence $\tilde{X}_1$ and $\tilde{X}_2$ are not homotopic, and hence not homeomorphic.