MAT 215A Take-Home Final Exam
Solutions

Note: All covering spaces are surjective maps.

1. Recall that the join $X * Y$ of two spaces X and Y is defined as $X \times Y \times I/\sim$ where $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$. Suppose that X and Y are path-connected and nonempty.

   (a) Prove that $X * Y$ is path-connected.

   SOLUTION. We can parametrize points in $X * Y$ by picking preimages in $X \times Y \times I$. Let (x, y, s) and (x', y', s') represent two points p, p' in $X * Y$, respectively.

   Because X, Y, I are path-connected, there exist paths

   - γ_X in X from $x \rightsquigarrow x'$
   - γ_Y from $y \rightsquigarrow y'$
   - γ_I from s to s' (e.g. the straight-line path).

   These induce paths in $X \times Y \times I$

   - $\alpha_X(t) = (\gamma_X(t), y, s)$ from $(x, y, s) \rightsquigarrow (x', y, s)$
   - $\alpha_Y(t) = (x', \gamma_Y(t), s)$ from $(x', y, s) \rightsquigarrow (x', y', s)$
   - $\alpha_I(t) = (x', y', \gamma_I(t))$ from $(x', y', s) \rightsquigarrow (x', y', s')$.

   These project to paths $\beta_X, \beta_Y, \beta_I$ in $X * Y$.

   Then $\beta_X \bullet \beta_Y \bullet \beta_I$ is a path in $X * Y$ from $p \rightsquigarrow p'$. Since every pair of points can be connected by a path, $X * Y$ is path-connected. $\square$

   (b) Prove that $X * Y$ is simply-connected.
   Hint: Use the Seifert-van Kampen theorem.

   SOLUTION.

   Let A be the image of $X \times Y \times \{0.5\}$ in $X * Y$. Since X and Y are path-connected, $A \cong X \times Y$ is also path-connected. (As in part (a), we may take a composition of paths in the X and Y components to move from one point to another.)

   Let U be the image of $X \times Y \times [0, 0.6)$ and V be the image of $X \times Y \times (0.4, 1]$. Together, these cover $X * Y$.

   Then U, V are open subsets of $X * Y$, where $U \cap V$ is the neighborhood (the homeomorphic image of) $X \times Y \times (0.4, 0.6)$ of A, which deformation retracts onto A, and is therefore path-connected.

Observe that U deformation retracts onto X, via the homotopy $f_t(x, y, s) = (x, y, (1-t)s)$. Similarly, V deformation retracts onto Y, via the homotopy $g_t(x, y, s) = (x, y, 1-(1-t)s)$.

Let $p_0 = (x_0, y_0, 0.5) \in A$ be a basepoint. Since $\pi_1(A, p_0) \cong \pi_1(X \times Y, p_0) \cong \pi_1(X, x_0) \times \pi_1(X, y_0)$, in order to use the Seifert–van Kampen theorem, it suffices to consider loops γ_X in $X \times \{y_0\} \subset A$ and γ_Y in $\{x_0\} \times Y \subset A$.

A nullhomotopy of γ_X can be constructed as follows. For $s \in [0.5, 1]$, let α_s be the path that travels from $(x_0, y_0, 0.5)$ to (x_0, y_0, s) in unit time, and let γ_X^s be the copy of γ_X in the slice $X \times Y \times \{s\}$ in $X * Y$ (where at $s = 1$, we only have a copy of Y). After reparametrizing, this is a homotopy from $\gamma_X \sim \alpha_1 \cdot \bar{\alpha}_1$, since γ_X^1 is a constant map. Compose this homotopy with the homotopy $\alpha_1 \bullet \bar{\alpha}_1 \simeq c_{p_0}$, the constant path at the basepoint.

We can similarly produce a nullhomotopy $\gamma_Y \simeq c_{p_0}$. Hence in the quotient of $\pi_1(U, p_0) * \pi_1(V, p_0)$ in the Seifert–van Kampen theorem, any element in $\pi_1(U, p_0) \cong \pi_1(X, x_0)$ or in $\pi_1(V, p_0) \cong \pi_1(Y, y_0)$ is identified with $[c_0]$. Hence $\pi_1(X * Y, p_0) \cong 1$. □

2. Let A be a path-connected subspace of a space X, with $x_0 \in A$. Show that the inclusion map $i : A \hookrightarrow X$ induces a surjective map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ if and only if any path with endpoints in A is homotopic (rel endpoints) to a path in A.

   Hint: If $\gamma \simeq \gamma'$ (rel endpoints), then $\alpha \cdot \gamma \cdot \beta \simeq \alpha \cdot \gamma' \cdot \beta$.

   Solution.

   (The symbol $\simeq$ always means 'homotopic rel endpoints' in the solution below.)

   The backwards implication is immediate: any loop γ representing a class in $\pi_1(X, x_0)$ can be homotoped to a path γ' in A, so $i([\gamma']) = [\gamma]$, so i_* is surjective.

   For the forward implication, let γ be a path with $\gamma(i) = a_i \in A$, $i = 0, 1$. Since A is path-connected, there exist paths α_i in A from $x_0 \in A$ to a_i, $i = 0, 1$. Let γ' be the path $\bar{\alpha}_0 \bullet (\alpha_0 \bullet \gamma \bullet \bar{\alpha}_1) \bullet \alpha_1$. Observe that (1) $\gamma'(i) = a_i$ for $i = 0, 1$, and (2) $\gamma' \simeq \gamma$.

   Since i_* is surjective, there exists some $[\beta] \in \pi_1(A, x_0)$ such that $i_*([\beta]) = [\alpha_0 \bullet \gamma \bullet \bar{\alpha}_1]$, where β is a loop completely contained in A. Therefore $\gamma \simeq \gamma' \simeq \bar{\alpha}_0 \bullet \beta \bullet \alpha_1$, a path entirely contained in A. □

3. Suppose X is a path-connected space with universal covering space $\tilde{X}$. Prove that if $\tilde{X}$ is compact, then $\pi_1(X)$ is finite.

   Solution.

   In this problem, students are allowed to assume that X is also locally path-connected.

   Let $p : \tilde{X} \to X$ be the (surjective) covering map from the universal cover.

   We claim that X is also compact. Let $\mathcal{C}$ be a cover of X consisting of evenly covered neighborhoods. Let $\tilde{\mathcal{C}}$ be the cover of $\tilde{X}$ consisting of homeomorphic preimages of $U \in \mathcal{C}$ under the covering map p. Since $\tilde{X}$ is compact, $\tilde{\mathcal{C}}$ admits a finite subcover $\tilde{\mathcal{C}}'$. Then $\mathcal{C}' := \{p(V) \mid V \in \tilde{\mathcal{C}}'\} \subset \mathcal{C}$ is a finite subcover of X.

   Now choose a cover C of X consisting of evenly covered neighborhoods such that exactly one open set $U \in C$ contains x_0. Let $\tilde{C}$ the induced cover of $\tilde{X}$ as described in the previous paragraph. Let $\tilde{C}'$ be finite subcover of $\tilde{C}$. Since lifts of U_0 are disjoint and also x_0 is uniquely covered by U_0 in C, every lift $\tilde{x}_0$ of x_0 is contained in a unique $V \in \tilde{C}'$. Since $|C'| < \infty$, there are only finitely many lifts $\tilde{x}_0$.

Recall that $\pi_1(X, x_0)$ acts as deck transformations on $\tilde{X}$. Namely, given a chosen $\tilde{x}_0 \in \tilde{X}$ lifting x_0, the elements $[\gamma] \in \pi_1(X, x_0)$ are in bijection with the lifts $\tilde{x}_0' \in p^{-1}(x_0)$, via $[\gamma] \mapsto \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the unique lift of γ starting at x_0.

Hence $|\pi_1(X, x_0)| = |p^{-1}(\tilde{x}_0)| < \infty$. □

4. Let X be a CW complex.

   (a) Let $\tilde{X}$ be a k-sheeted cover of X. Show that $\tilde{X}$ also has a CW structure, and that $\chi(\tilde{X}) = k\chi(X)$, where χ denotes Euler characteristic.

   Solution.

   Suppose we have a CW structure on X consisting of cells $\{e^n_\alpha\}_{\alpha \in A}$, where e^n_α has dimension n and A is some indexing set.

   Let $p : \tilde{X} \to X$ be a k-sheeted covering. A CW structure on $\tilde{X}$ can be described as follows. Note that π_1 of any D^n is trivial, so the lifting criterion is satisfied.

   - The 0-cells of $\tilde{X}$ are the points that are preimages of the 0-cells of X.
   - The 1-cells are given by (all) the lifts of the maps $\gamma : D^1 = I \to X$ defining the 1-cells of X; clearly the endpoints $\tilde{\gamma}(0), \tilde{\gamma}(1)$ are 0-cells of $\tilde{X}$, since $\gamma(0), \gamma(1) \in X^0$.
   - Similarly, the n-cells are given by all the lifts of the maps $f : D^n \to X$ defining the n-cells of X, and $\tilde{f}(\partial D^n) \subset \tilde{X}^{n-1}$ because $f(\partial D^n) \subset X^{n-1}$.

   By the unique lifting property, distinct lifts of $e^n_\alpha : D^n \to X^n$ have disjoint images. Since every point in $\tilde{X}$ projects via p to some cell in X, the above is a CW structure on $\tilde{X}$.

   For each cell e^n_α, pick a point in p_α the interior. (For example, viewing e^n_α as a map $D^n \to X$, we may choose the image of the origin.) Since p is k-sheeted, there are k lifts of p_α, so there are exactly k cells in $\tilde{X}$ that lift e^n_α. Since

   $$\chi(X) = \sum_{n \geq 0} (-1)^n \#(n\text{-cells in } X),$$

   we have

   $$\chi(\tilde{X}) = \sum_{n \geq 0} (-1)^n \#(n\text{-cells in } \tilde{X}) = \sum_{n \geq 0} (-1)^n k \cdot \#(n\text{-cells in } X) = k\chi(X)$$

   (when $\chi(X)$ is defined). □

   (b) Show that if $f : \mathbb{RP}^{2n} \to X$ ($n \geq 1$) is a covering map, then f is a homeomorphism.

   Solution. Since $\mathbb{RP}^{2n}$ has one cell in each of its $2n+1$ dimensions, $\chi(\mathbb{RP}^{2n}) = 1$. Since $\mathbb{RP}^{2n}$ is compact (and path-connected), by Question 3 we know that $\pi_1(X)$ (no need to specify basepoint) is finite, so the maximum number of sheets a cover can have is $|\pi_1(X)| < \infty$. By part (a), $1 = \chi(\mathbb{RP}^{2n}) = k\chi(X)$ for some k, and since $\chi(X) \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, we must have $k = 1$. Hence f is a 1-sheeted covering space, i.e. a homeomorphism. □