Commutative Diagrams

Melissa Zhang

Mat 215A Fall 2025

We use commutative diagrams to visualize equations involving functions. These can be drawn for any category, but you may first just think about functions between sets.

1 Basics

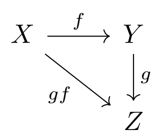
We visualize a function $f : X \to Y$ by drawing the following diagram:

$$X \xrightarrow{f} Y$$

If $f : X \to Y$ and $g : Y \to Z$ are composable functions, we depict the composition $g \circ f$ (also written gf) by concatenating their diagrams:

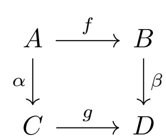
$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

The three functions f, g, and $g \circ f$ are related by the following *commutative diagram*:



This means that if you pick any element $x \in X$, $g(f(x)) = gf(x)$. (This looks tautological because it is.)

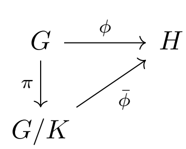
We can use diagrams to write down more complicated relationships among functions. For example, the diagram



is a visualization of the equation $\beta f = g\alpha$.

2 Factoring through

Now suppose we are working with groups and group homomorphisms. Let $\phi : G \to H$ be a homomorphism with kernel $K = \ker \phi$. Since everything in K is sent to 1_H, there is a well-defined map $\bar{\phi} : G/K \to H$ induced by ϕ, so we have a commutative diagram



where π is the quotient map. In fact, for any $N \subset \ker \phi$ where $N \lhd G$, for same reason we have a diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle \pi}\big\downarrow & \nearrow_{\bar{\phi}} & \\ G/N & & \end{array}$$

because $\phi(N) = 1_H$. In this case we say that ϕ *factors through* the quotient group G/N.

In the category of topological spaces, we have the same concept. If a map $f : X \to Y$ is constant on some subspace $A \subset X$, then f factors through the quotient space X/A, i.e. we have the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \pi}\big\downarrow & \nearrow_{\bar{f}} & \\ X/A & & \end{array}$$

where π is the quotient map. We say that f *factors through* X/A, and that the map f factors as the composition $\bar{f}\pi = f$.

Exercise 2.1. Check for yourself that $\bar{\phi}$ and $\bar{f}$ above are actually well-defined.

Remark 2.2. There are more situations where we say a map 'factors through' something. In the above setting we could also say that 'f factors through $\bar{f}$'.

If we are in the setting of the diagram

$$\begin{array}{ccc} & & Y \\ & {}^{\tilde{f}}\nearrow & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & B \end{array}$$

we would say that f factors through $\tilde{f}$. In this case, we call $\tilde{f}$ a *lift* of f.

In general, we say 'factors through' to indicate that a map factors into a composition of two maps.

Exercise 2.3. Let $p : \mathbb{Z} \to \mathbb{Z}/10$ be the 'mod 10' quotient map, and let $q : \mathbb{Z} \to \mathbb{Z}/5$ be the 'mod 5' quotient map. Show that q factors through p.