MAT 215B HW01 Exercise 3 Sample Solution

Melissa Zhang

4/15/2025

Disclaimer: I don't claim to be "the best" at writing. There are also differences in writing culture among different fields.

This sample solution is provided as a rough guideline for how much detail I personally believe is needed in a (calculational / constructive) proof in this course. Writing in less detail than this might be risky (but if you're good at writing clearly and concisely I won't complain). You are free to write more, to some reasonable degree, if it helps you sort out solutions for yourself.

Remember, ultimately, the purpose of homework is for you to check and solidify your understanding of the course material.

Exercise 1

Find a Δ-complex structure for the n-dimensional sphere

$$S^n = \{p \in \mathbb{R}^{n+1} \mid |p| = 1\}$$

where $n \geq 0$.

You may assume without proof that there is a canonical homeomorphism from the n-simplex Δ^n to the n-disk

$$D^n = \{q \in \mathbb{R}^n \mid |q| \leq 1\}.$$

Make sure you verify all the properties of a Δ-complex!

Solution.

Consider the canonical homeomorphism $\Delta^{n+1} \to D^{n+1}$, and let

$$f : \partial\Delta^{n+1} \to S^n$$

denote the restricted homeomorphism on the boundaries of both the domain and target spaces of the canonical homeomorphism.

Observe that $\partial\Delta^{n+1}$ already has a Δ-complex structure, given by the collection of inclusion maps $\{\iota_\alpha : \Delta^{n(\alpha)} \to \partial\Delta^{n+1}\}$ for each face E_α of Δ^{n+1} where E_α has dimension $\leq n$.

⋄

We claim that the corresponding collection of maps $\{\sigma_\alpha := f \circ \iota_\alpha\}$ is a Δ-complex structure on S^n. We check the three conditions Δ-1, Δ-2, and Δ-3:

(Δ-1) Since $\{\iota_\alpha\}$ is a Δ-complex structure on $\partial\Delta^{n+1}$, ι_α is an injective continuous map on $int(\Delta^{n(\alpha)})$. Since f is a homeomorphism $f \circ \iota_\alpha$ is also injective on $int(\Delta^{n(\alpha)})$ for all α. Second, any point $s \in S^n$ is $f(p)$ for some $p \in \partial\Delta^{n+1}$; since p is in the image of exactly one restriction $\iota_\alpha|_{int(\Delta^{n(\alpha)})}$, so is s (as f is a homeomorphism).

(Δ-2) Given σ_β and a face F of $\Delta^{n(\beta)}$, the restriction $\sigma_\beta|_F$ is equal to $f \circ \iota_\beta|_F$. Since the $\{\iota_\alpha\}$ give a Δ-complex structure on $\partial\Delta^{n+1}$, there is some ι_γ equal to the restriction. Then $\sigma_\gamma = \sigma_\beta|_F$.

(Δ-3) Since f is a homemorphism, $U \subset S^n$ is open iff $f^{-1}(U) \subset \partial\Delta^{n+1}$ is open, which is true iff $\iota_\alpha^{-1}(f^{-1}(U))$ is open because ι_α satisfies (Δ-3). But $\iota_\alpha^{-1}(f^{-1}(U)) = (f \circ \iota_\alpha)^{-1}(U) = \sigma_\alpha^{-1}(U)$, as desired.

For the first homework, we are being very careful about simplicial / Δ-complex structures because that is the focus of the chapter. Later on in the course, and in your paper-writing career, it would be acceptable to replace the remainder of the proof after the diamond $\diamond$ with the following statement:

"The claim that the collection $\{\sigma_\alpha\}$ is a Δ-complex structure follows directly from the fact that the collection $\{\iota_\alpha\}$ is a Δ-complex structure on $\partial\Delta^{n+1}$, and that f is a homeomorphism."

However, on this homework, I asked you to specifically check the conditions for being a Δ-complex structure.