MAT 215B HW01

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Due Friday, 4/11/25 at 9:00 pm on Gradescope

Instructions The exercises below have been categorized based on what you're supposed to get out of them. Please submit your completed solutions to Gradescope by the due date shown above.

- Your solution **must be typeset**, but you may include handdrawn figures. LaTeX is strongly preferred of course, for your own benefit, but you may use different typesetting software if you'd like. Give yourself enough time to typeset your work (which always involves re-thinking your proofs and making them clearer)!
- Ideally, you should try all of them and then come to office hours for help on the ones you're stuck on. However, if you haven't gotten to a problem yet by the time office hours are held, I still encourage you to come to office hours.
- No extensions will be given; your lowest homework grade will be dropped at the end of the quarter. If you are behind on homework, I encourage you to submit a few solutions rather than no solutions!
- If you find any errors, typos, or unclear statements, feel free to email me to let me know.

Quick Exercises

Quick exercises are graded for completion. Your solution should still be professionally written (e.g., in full sentences) but you may choose to write just enough to show that you have completed the exercise for yourself.

Exercise 1

Consider the 3-simplex Δ^3.

(a) Identify and draw all faces of Δ^3. Determine the orientations of all the faces of dimensions 1 and 2 (i.e. the orientations induced by the total ordering on vertices).

(b) Let e^3 denote the interior of the 3-simplex. Determine the orientations on the dimension-2 faces induced by the *boundary orientation* of e^3.

(c) Compare the orientations on the dimension-2 faces you got in parts (a) and (b): note which subset of edges have different v.s. the same orientations in (a) and (b). Can you describe these two sets?

Exercise 2

(Hatcher §2.1, exercise 1) What familiar space is the quotient Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of the vertices?

Computational Exercises

Computational exercises are specifically intended for you to practice a computation or algorithm. Your solution should be complete and rigorous, but not egregiously long; you should make reasonable choices about what which 'obvious' steps to omit in your writeup.

Exercise 3

Find a Δ-complex structure for the n-dimensional sphere

$$S^n = \{p \in \mathbb{R}^{n+1} \mid |p| = 1\}$$

where $n \geq 0$.

You may assume without proof that there is a canonical homeomorphism from the n-simplex Δ^n to the n-disk

$$D^n = \{q \in \mathbb{R}^n \mid |q| \leq 1\}.$$

Make sure you verify all the properties of a Δ-complex!

Exercise 4

Find a Δ-complex structure for the genus g closed, connected, orientable ('cco') surface Σ_g, which can be constructed by identifying edges of the $4g$-gon according to the word

$$\prod_{i=1}^{g} [a_i, b_i] = \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$$

where $g \geq 1$. (See Hatcher page 5 for a figure.)

Exercise 5

Find a Δ-complex structure for the nonorientable genus k closed, connected, nonorientable surface P_γ, which can be constructed by identifying the edges of the $2k$-gon according to the word

$$\prod_{i=1}^{\gamma} a_i^2 = a_1^2 a_2^2 \cdots a_\gamma^2$$

where $\gamma \geq 1$.

Rigorous Exercises

Rigorous exercises generally involve proving a statement. You should give a full proof as your solution. Some of these problems may also be somewhat computational, depending on the topic.

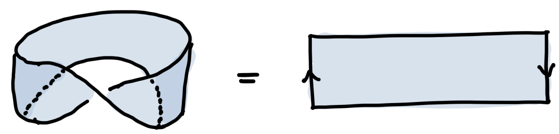
Exercise 6

In class, we gave a Δ-complex structure to the torus T. Compute the simplicial homology of T explicitly.

Now is a great time to learn to use the `tikzcd` package!

Exercise 7

The Möbius band M is a famous compact nonorientable surface that can be built as follows:



(a) Find a Δ-complex structure on M.

(b) Use the Δ-complex structure from (a) to compute the simplicial homology of M.

Exercise 8

(Hatcher §2.1, exercise 9) Consider the Δ-complex X obtained from Δ^n by identifying all faces of the same dimension. Note that X has a single k-simplex for each dimension $k \leq n$.

Compute the homology groups of the Δ-complex X.

How many of the spaces in this family can you identify, up to homeomorphism?

Optional Exercises

Optional problems are just for you to think about. We will not be relying on these results too much, but it is good to know these results. These are not graded; I probably won't look at your solution to these too carefully, or maybe at all.

Exercise 9

Put a Δ-complex structure on the [Hedgehog space] of cardinality κ.

Exercise 10

Show that the dunce cap is contractible.

Exercise 11

Let $(\mathcal{C}, \partial)$ be a chain complex as shown below:

$$\cdots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

(Recall that $\mathcal{C} = \bigoplus_i C_i$.) Denote the *total differential* by $\partial : \mathcal{C} \to \mathcal{C}$, $\partial = \sum_i \partial_i$. If this notation doesn't make complete sense, you can think of ∂ as a giant matrix with infinitely many rows and columns. Prove that the condition

$$\partial \circ \partial = 0$$

(i.e. $\partial^2 = 0$) is equivalent to the condition

$$\partial_{i-1} \circ \partial_i = 0 \qquad \text{for all } i.$$