MAT 215B HW02

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Due Friday, 4/18/25 at 9:00 pm on Gradescope

Instructions The exercises below have been categorized based on what you're supposed to get out of them. Please submit your completed solutions to Gradescope by the due date shown above.

- Your solution **must be typeset**, but you may include handdrawn figures. LaTeX is strongly preferred of course, for your own benefit, but you may use different typesetting software if you'd like. Give yourself enough time to typeset your work (which always involves re-thinking your proofs and making them clearer)!
- Ideally, you should try all of them and then come to office hours for help on the ones you're stuck on. However, if you haven't gotten to a problem yet by the time office hours are held, I still encourage you to come to office hours.
- No extensions will be given; your lowest homework grade will be dropped at the end of the quarter. If you are behind on homework, I encourage you to submit a few solutions rather than no solutions!
- If you find any errors, typos, or unclear statements, feel free to email me to let me know.

Quick Exercises

Quick exercises are graded for completion. Your solution should still be professionally written (e.g., in full sentences) but you may choose to write just enough to show that you have completed the exercise for yourself.

Exercise 1

Explain why $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$ for any space $X \neq \emptyset$.

You may use the following lemma:

Lemma. If $A \xrightarrow{\alpha} \mathbb{Z}$ is a surjective homomorphism of $\mathbb{Z}$-modules, then $A \cong \ker \alpha \oplus \mathbb{Z}$.

This fact is implicitly used in the text. This is however not necessarily true if we replace $\mathbb{Z}$ with other $\mathbb{Z}$-modules! For example, we have a surjective map $\mathbb{Z}/4\mathbb{Z} \xrightarrow{f} \mathbb{Z}/2\mathbb{Z}$ with $\ker(f) \cong \mathbb{Z}/2\mathbb{Z}$. However, $\mathbb{Z}/4\mathbb{Z}$ is *not* isomorphic to the Klein-4 group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (which has no order-4 element).

Computational Exercises

Exercise 2

It is easy to understand maps (e.g. singular simplicies) into a single-point space $X = \{*\}$. Prove that $\widetilde{H}_n(X) = 0$ for all n by writing down the singular chain complex and computing the homology. This is in the text as well. Use your own words.

Rigorous Exercises

Rigorous exercises generally involve proving a statement. You should give a full proof as your solution. Some of these problems may also be somewhat computational, depending on the topic.

Exercise 3

(Hatcher §2.1 Ex 11) Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Exercise 4

For an exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$$

show that $C = 0$ iff α is surjective and δ is injective.

This is the first half of §2.1 Ex 15. The second half of the exercise states, "Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n." We will talk about this in class.

Exercise 5

This exercise introduces the mapping cone construction in homological algebra. Let $(\mathcal{C}, \partial)$ and $(\mathcal{C}', \partial')$ be two chain complexes, and let $f : \mathcal{C} \to \mathcal{C}'$ be a chain map.

We define the *mapping cone of* f as the chain complex $(\mathrm{Cone}(f), D)$ given by the following:

- $\mathrm{Cone}(f)_n = C_n \oplus C'_{n+1}$
- $D_n = \begin{pmatrix} -\partial_n & 0 \\ f_n & \partial'_{n+1} \end{pmatrix}$

(a) Prove that $\mathrm{Cone}(f)$ is indeed a chain complex.

(b) Suppose the chain map f induces an isomorphism on homology, i.e. $(f_*)_n : H_n(\mathcal{C}) \to H_n(\mathcal{C}')$ is an isomorphism for all n. Prove that $\mathrm{Cone}(f)$ is *acyclic*, i.e. $H_n(\mathrm{Cone}(f)) = 0$ for all n.

Exercise 6

This problem will be moved to the next homework set and will be modified to allow you to use the LES for homology groups.

For a space X, let SX denote the *free suspension* of X, which is thought of as two cones CX of X glued together along their boundaries, which are homeomorphic to X. Show that $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ for all n by constructing an explicit chain map $s = \{s_n : C_n(X) \to C_{n+1}(SX)\}$ that induces isomorphisms $\widetilde{H}_n(X) \xrightarrow{\cong} \widetilde{H}_{n+1}(SX)$.

(Ungraded) Optional Exercises

Exercise 7

Describe a cell decomposition for S^n $(n \geq 0)$ consisting of two cells in each dimension.

Recall that

$$S^n = \{p = (x_0, x_1, \ldots, x_n) \subset \mathbb{R}^{n+1} \mid |p| = 1.$$