MAT 215B HW03

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Due Friday, 4/25/25 at 9:00 pm on Gradescope

Instructions The exercises below have been categorized based on what you're supposed to get out of them. Please submit your completed solutions to Gradescope by the due date shown above.

- Your solution **must be typeset**, but you may include handdrawn figures. LaTeX is strongly preferred of course, for your own benefit, but you may use different typesetting software if you'd like. Give yourself enough time to typeset your work (which always involves re-thinking your proofs and making them clearer)!
- Ideally, you should try all of them and then come to office hours for help on the ones you're stuck on. However, if you haven't gotten to a problem yet by the time office hours are held, I still encourage you to come to office hours.
- No extensions will be given; your lowest homework grade will be dropped at the end of the quarter. If you are behind on homework, I encourage you to submit a few solutions rather than no solutions!
- If you find any errors, typos, or unclear statements, feel free to email me to let me know.

Quick Exercises

Quick exercises are graded for completion. Your solution should still be professionally written (e.g., in full sentences) but you may choose to write just enough to show that you have completed the exercise for yourself.

Exercise 1

Prove Corollary 2.11 by using Theorem 2.10.

Exercise 2

Use the LES for the homology of a pair to prove that

$$H_\bullet(X, \{x_0\}) \cong \widetilde{H}_\bullet(X)$$

where x_0 is a point in $X \neq \emptyset$.

Exercise 3

This exercise isn't exactly quick, but the solution is in the book (Hatcher, bottom of page 129), and the point is that you should work through it. This will be graded for completion, and you do not need to type up your notes on this proof if writing by hand is the way you like to check proofs.

We will need the following lemma in our proof that $H^\Delta_\bullet(X) \cong H_\bullet(X)$.

Lemma. (The Five-Lemma) Consider a commutative diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \varepsilon \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E' \end{array}$$

where the two rows are exact. If α, β, δ, and ε are isomorphisms, then γ is also an isomorphism.

In this exercise you will prove this lemma, along with some more general statements.

(a) Suppose β and δ are surjective, and ε is injective. Prove that γ is surjective.

(b) Suppose β and δ are injective, and α is surjective. Prove that γ is injective.

(c) Prove the Five-Lemma.

Computational Exercises

Computational exercises are specifically intended for you to practice a computation or algorithm. Your solution should be complete and rigorous, but not egregiously long; you should make reasonable choices about what which 'obvious' steps to omit in your writeup.

Exercise 4

(a) Compute $H_\bullet(X, A)$ where $X = S^2$ and A is a set of k points in X.

(b) Compute $H_\bullet(X, A)$ where $X = S^1 \times S^1$ and A is one of the S^1's. In other words, X is a torus and A is a meridian.

Rigorous Exercises

Rigorous exercises generally involve proving a statement. You should give a full proof as your solution. Some of these problems may also be somewhat computational, depending on the topic.

Exercise 5

At the end of our proof that singular homology is homotopy invariant, we relied on the following algebraic fact.

Let $\mathcal{A}$ and $\mathcal{B}$ be chain complexes:

$$\mathcal{A} = \cdots \to A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} A_{n-2} \to \cdots$$

$$\mathcal{B} = \cdots \to B_n \xrightarrow{\beta_n} B_{n-1} \xrightarrow{\beta_{n-1}} B_{n-2} \to \cdots$$

Suppose $f, g : \mathcal{A} \to \mathcal{B}$ are chain maps satisfying $g - f = \partial^{\mathcal{B}} h + h \partial^{\mathcal{A}}$ for some chain homotopy h. Prove that $f_* = g_* : H_\bullet(\mathcal{A}) \to H_\bullet(\mathcal{B})$.

Exercise 6

Suppose we have two pairs of spaces $A \subseteq X$ and $B \subseteq Y$, and a map $f : X \to Y$ such that $f(A) \subseteq B$. Furthermore, suppose that f induces a homotopy equivalence from X to Y, and $f|_A$ induces a homotopy equivalence from A to B. Prove that $f_* : H_\bullet(X, A) \to H_\bullet(Y, B)$ is an isomorphism.

Hint: Use the LES on homology of a pair together with the Five-Lemma.

Exercise 7

(Hatcher §2.1 Ex 20)

Use the LES of a pair to show that $\widetilde{H}_n(X) \cong \widetilde{H}_{n+1}(SX)$ for all n, where SX is the (free) suspension of X. See page 8 of Hatcher for the definition of SX.

You may assume that (CX, X) is a good pair, where CX is $X \times [0,1]/(x,1)$ and X is identified with $X \times \{0\}$.