LECTURES 1, 2, 3 (+ DISCUSSION 1) 3.31.25 — 4.4.25

ie well-defined assignment to X

Let X be a top space. We wish to develop an 'invariant' of X:

combinatorial description of X $\rightsquigarrow$ $\mathbb{Z}$-graded $\mathbb{Z}$-module

one $\mathbb{Z}$-module per dimension $d \geq 0$, actually

called the "homology of X".

Next two weeks: Discuss multiple (equivalent) definitions of "the homology":

① simplicial homology

- computed from a Δ-complex structure on X

② singular homology

- less straightforward / combinatorial but more robust

Sometimes properties are easier to prove when the construction is more robust...

③ CW homology

- computed from a CW-complex structure on X
- most computationally efficient, most practical and the one you really ought to learn to compute!

④* Čech cohomology

- "local-to-global" flavor, eg. sheaf theoretic
- we may only touch upon this briefly

First things first

This week:

- how to construct/deconstruct Δ-complexes, many examples to keep in your pocket
- translation to algebra, i.e. the simplicial chain cpx

* Along the way, will introduce language of homotopy theorists i.e. viewing these simplicial structures purely "combinatorially" ← discrete data, ordered sets, ...

later on (next week)

- properties, structure theorems, cut + paste computation methods...
- equivalences among the different constructions

* * *

In these notes:

- "eg" = example
- "ex" = exercise
- 🟩 = defining this term
- underline for emphasis.

* These are my lecture notes, and the exercises are not fleshed out here; just reminders to self of ideas.

Simplicies plural of "simplex"
basic building blocks for (some) spaces

defn The **n-simplex Δ^n** is the convex hull of the n+1 points $\{\underbrace{v_0, \dots, v_n}\} = \{(1,0,\dots,0), \dots, (0,\dots,0,1)\}$

- corr. to the standard basis vectors $e_1, \dots, e_n$
- v_i for "vertices"

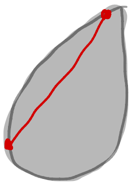
$$\Delta^n := \Big\{ \underbrace{(t_0, \dots, t_n)}_{= \sum_{i=0}^{n} t_i v_i \text{ since our } v_i \text{ are standard!}} \in \mathbb{R}^{n+1} \;\Big|\; \sum_i t_i = 1, \quad t_i \geq 0 \quad \forall i \Big\}$$

Aside The convex hull of a set of points $P = \{p_1, \dots, p_k\}$ in $\mathbb{R}^N$ is the smallest convex set containing all points in P

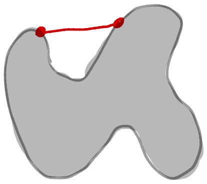
For any $x, y \in S$, all points on the line b/w x and y, are also in S

Some examples in $\mathbb{R}^2$:

Convex:



Not convex:



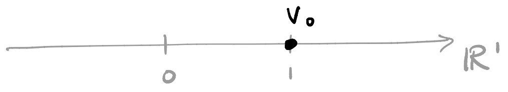
This is equivalent to the set of all linear combinations of points $p_i \in P$ where the nonnegative weights t_i add up to 100%:

$$\text{Hull}(P) = \Big\{ \sum_{i=1}^{k} t_i p_i \;\Big|\; \sum_{i=1}^{k} t_i = 1, \quad t_i \geq 0 \quad \forall i \Big\}$$

eg. The first few simplicies :

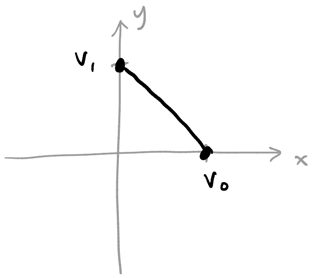
(geometrically) (abstractly / combinatorially)

$\Delta^0 =$



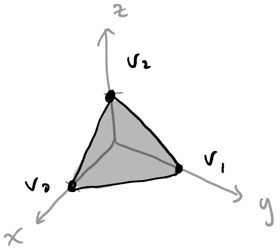
the ordered set $[0] = \{0\}$. actually (0) b/c ordered..

$\Delta^1 =$



the ordered set $[1] = \{0,1\}$ ie $(0,1)$

$\Delta^2 =$

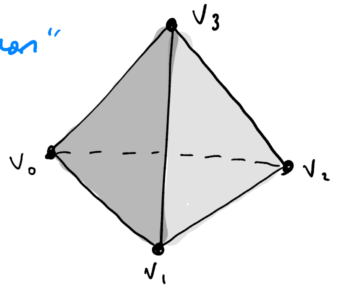


the ordered set $[2] = \{0,1,2\}$ (ie $(0,1,2)$)

$\Delta^3 =$ . . . well it looks like this, but embedded in $\mathbb{R}^4$:

the ordered set $[3]$

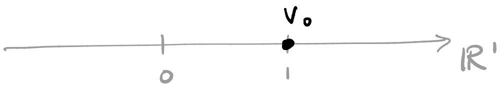
"tetrahedron"



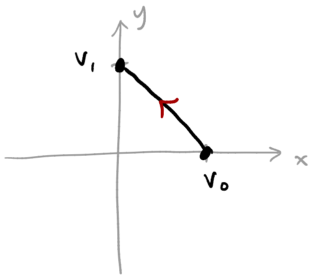
Rmk. Δ^n is an n-diml manifold, ie for any point in the interior, a nbhd looks like $\mathbb{R}^n$.

The ordering on the vertices determines an <u>orientation</u> on all the faces of a simplex :

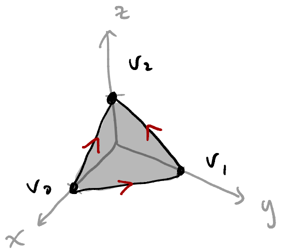
$\Delta^0 =$



$\Delta^1 =$

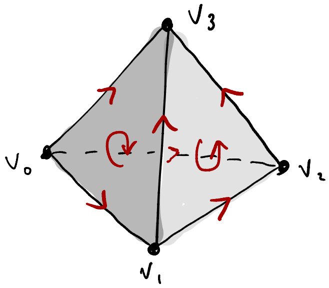


$\Delta^2 =$

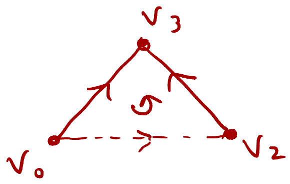


Interior orientation: ↺ ccw

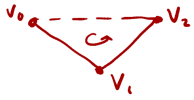
$\Delta^3 =$



back orientation :



bottom orientation :



These orientations are very important when we convert to algebra!

notation We capture all the abstract data of the n-simplex by the ordered set

$$[n] := \{0, 1, 2, \dots, n\}$$

maybe one likes to write $(0,1,\dots,n)$ but I'm not thinking of these as coordinates...

with the total ordering $\leq$.

⚠ Some notation warnings:

- In CS, $[n]$ usually means $\{1, 2, \dots, n\}$
- In homological algebra, $[n]$ means degree shift of n
- In quantum algebra, $[n]$ is a quantum integer.

defn A face (really, "subsimplex") of a simplex Δ^n is the subsimplex with vertices a nonempty subset of $\{v_0, \dots, v_n\}$.

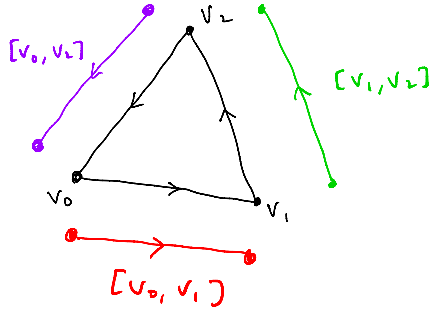
- A k-dim'l face of Δ^n (where $0 \leq k \leq n$) is a subsimplex homeomorphic to Δ^k
- Hatcher's notation: the subsimplex with vertices

  $v_{i_1}, v_{i_2}, \dots, v_{i_k}$ with $i_1 < i_2 < \dots < i_k$

  is written $[v_{i_1}, v_{i_1}, v_{i_2}, \dots, v_{i_k}]$
- A k-dim'l face of Δ^n corresponds to an order-preserving map $[k] \longrightarrow [n]$.

eg. Δ^2 has 3 codimension-1 (ie dimension 1) faces:

If X is n dim'l,
and $S \subset X$ is d dim'l,
the codim of S is $n-d$.



These can be written using "delete" notation:

- $[v_1, v_2] = [\hat{v}_0, v_1, v_2]$
- $[v_0, v_2] = [v_0, \hat{v}_1, v_2]$
- $[v_1, v_2] = [v_0, v_1, \hat{v}_2]$

These also correspond to the 3 order-preserving maps

$[1] \longrightarrow [2]$:

- $[v_1, v_2] = [\hat{v}_0, v_1, v_2]$: $0 \mapsto 1$, $1 \mapsto 2$
- $[v_0, v_2] = [v_0, \hat{v}_1, v_2]$: $0 \mapsto 0$, $1 \mapsto 2$
- $[v_1, v_2] = [v_0, v_1, \hat{v}_2]$: $0 \mapsto 0$, $1 \mapsto 1$.

ex. "ex" = exercise in my notes; "eg" = example

(a) How many faces of dimension k are there in Δ^n?

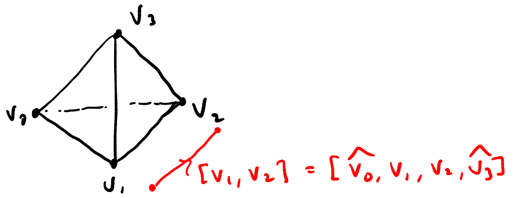
$\binom{n+1}{k+1}$ faces. You keep $k+1$ vertices.

(b) How many faces does Δ^n have total?

nonempty subsets of $[n] = 2^{n+1} - 1$.

ex. Identify and draw all faces of Δ^3, w/ orientation for dim 1 and 2 faces.

eg.



$[v_1, v_2] = [\hat{v}_0, v_1, v_2, \hat{v}_3]$

Δ-complexes (soft intro)

informal defn

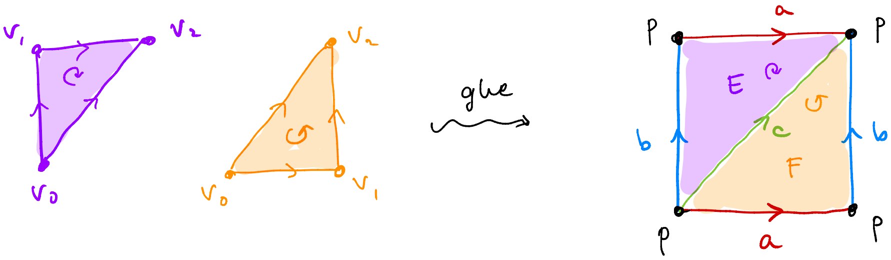
A **Δ-complex** is a quotient space of a collection of disjoint simplicies obtained by identifying faces in the order-preserving way

eg S^1 = the circle

There are many different ways to build S^1!

eg. S^2 the sphere

eg. the torus $T = S^1 \times S^1$:



Here is the formal definition:

defn. Let X be a topological space.

A Δ-complex structure on X is a collection of maps

$\{\sigma_\alpha : \Delta^{n(\alpha)} \longrightarrow X\}_{\alpha \in A}$ such that

$\boxed{\Delta\text{-}1}$ $\forall\ \alpha \in A$, $\sigma_\alpha \big|_{int(\Delta^{n(\alpha)})}$ is injective

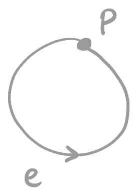
and each point in X is in the image of exactly one such restriction

* caveat For $S \subset \mathbb{R}^d$, the interior $int(S)$ of S is defined to be the union of all open subsets U of $\mathbb{R}^d$ contained in S: $int(S) = \bigcup \{U \subset \mathbb{R}^n \mid U \subseteq S\}$.

Since Δ^0 is a single point in $\mathbb{R}^1$, $int(\Delta^0) = \emptyset \ldots$

So, for the purposes of this definition, say "$int(\Delta^0) = \Delta^0$".

eg. $X = S^1$



point p is in the image of $\sigma_p : \Delta^0 \longrightarrow X$

but not in $\sigma_e \big|_{int(\Delta^1)} : int(\Delta^1) \longrightarrow X$.

$\boxed{\Delta\text{-}2}$ Each restriction $\sigma_\alpha \big|_{\text{face } F \text{ of } \Delta^{n(\alpha)}}$ where $F \cong \Delta^k$

is equal to some map $\sigma_\beta : \Delta^k \longrightarrow X$

↑ Here we need to identify $\Delta^{n(\beta)} = \Delta^k$, the src of σ_β, with the face F of $\Delta^{n(\alpha)}$ in the order preserving way.

⚠ There are geometric concerns about parametrization.
We claim there is an "obvious" canonical way to identify F with Δ^k in the order-preserving way and speak no more of this.

$\boxed{\Delta\text{-}3}$ $U \subset X$ is open $\iff$ $\forall \alpha \in A$, $\sigma_\alpha^{-1}(U)$ is open. //

Rmk. (Casual interpretation of the requirements)

(Δ-1) tells us that X is built specifically and entirely from these pieces.

(Δ-2) tells us that if a simplex Δ^n is used to build X, then all its faces (subsimplicies) are also part of the construction, ie they are domains of other maps in the collection.

(Δ-3) tells us that the quotient topology on the constructed space agrees with the topology of the originally given space X.

rmk. (Aside)

① Cardinality of A need not be finite, but we will mostly work with finite data

ex. (Remark about cardinality of A). Put a Δ-complex structure on the Hedgehog space for cardinality K.

② Not all spaces have a Δ-structure!

Fact A closed 4-mfld is triangulable iff it is smoothable.

I have not provided a definition here, but it is equivalent to being the geom. realization of a simplicial set.

eg. The E^8 manifold (!! It's even a manifold!) is not triangulable.

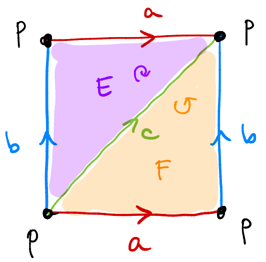
③ However, **Fact** every manifold is homotopic (but not necessarily homeomorphic) to a CW complex. ☺

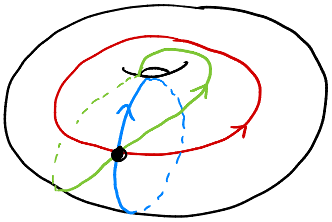
↳ we'll get there.

Examples

Keep these in your pocket!

eg. Torus $T = S^1 \times S^1$ (more formally now)





The collection of maps: $|A| = 6$ maps

- $\sigma_p : \Delta^0 \longrightarrow X$ sends $v_0 \longmapsto p$
- $\sigma_a : \Delta^1 \longrightarrow X$ sends $[v_0, v_1]$ to a

  $\sigma_b : \Delta^1 \longrightarrow X$ sends $[v_0, v_1]$ to b

  $\sigma_c : \Delta^1 \longrightarrow X$ sends $[v_0, v_1]$ to c

  } via the "canonical" (i.e. obvious) homeomorphism

All three of $\sigma_a, \sigma_b, \sigma_c$ send the endpoints $v_0, v_1 \longmapsto p$

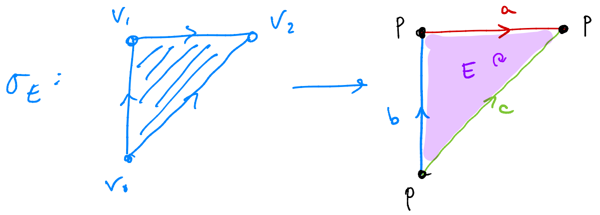
- $\sigma_E : \Delta^2 \longrightarrow X$ sends $[v_0, v_1, v_2] \longrightarrow E$

  and in particular sends

  $[v_0, v_1] \longrightarrow b$

  $[v_1, v_2] \longrightarrow a$

  $[v_0, v_2] \longrightarrow c$



  and sends $[v_0], [v_1], [v_2] \longrightarrow p$.

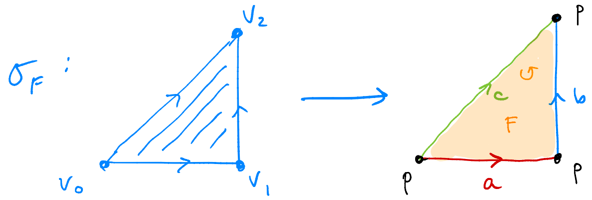
- σ_F $\Delta^2 \longrightarrow X$ sends $[v_0, v_1, v_2] \longrightarrow E$

  and in particular sends

  $[v_0, v_1] \longrightarrow a$

  $[v_1, v_2] \longrightarrow b$

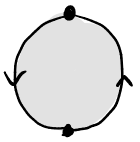
  $[v_0, v_2] \longrightarrow c$



  and sends $[v_0], [v_1], [v_2] \longrightarrow p$.

eg. $\mathbb{R}P^2$: 2-dim'l real projective space
$= S^2/\sim$ where $\sim$ identifies antipodal points

We often think of $\mathbb{R}P^2$ topologically as the space we get from identifying two sides of a bigon:

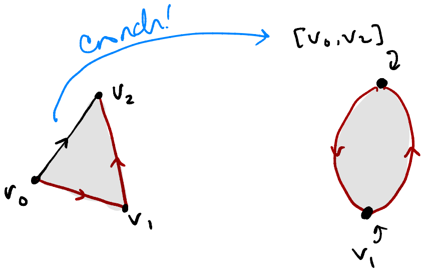


⚠ NOT a dumpling.
Note the orientations.

This is not a Δ-complex structure. Here's a thought process:

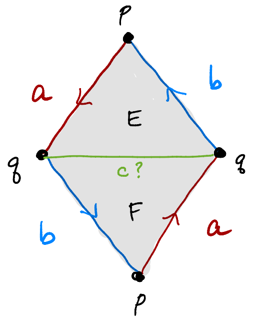
Attempt 1: Bad

Let's try a single Δ^2:



Violates
Δ-1
injectivity.

Attempt 2: better, but still not legal



Now the path ba is mapped to the "seam"

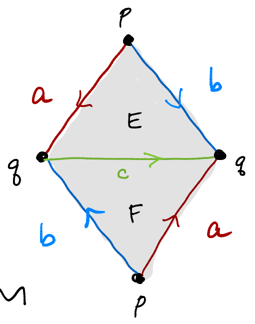
There is no way to orient c so that

$$\sigma_E\Big|_{\text{face "c"}} = \sigma_F\Big|_{\text{face "c"}}.$$

Either choice violates Δ-2.

Attempt 3 slight modification, now good

just reverse the orientation of b:



Verify that now:

$$\sigma_E\Big|_{[v_1,v_2]} = \sigma_c = \sigma_F\Big|_{[v_0,v_1]}$$

I've chosen an orientation on c.

ex. Build (Δ-) simplicial structures for

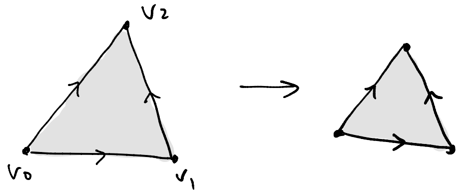
(a) spheres of all dims

(b) cco surfaces   cco = closed, cntd, orientable

(c) cc, non-orientable surfaces

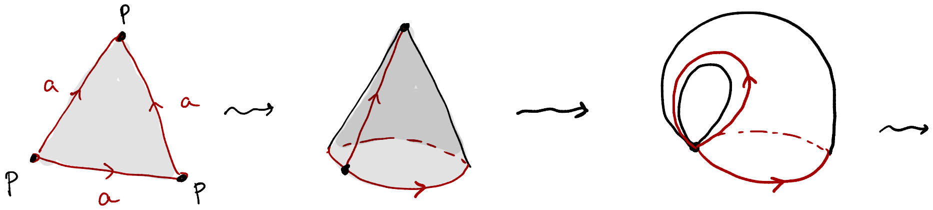
(d) n-torus

eg. "Dunce cap"   What can I make with just one Δ^2 (+ all faces)?



i.e. identify all 3 sides, according to these orientations!

One can literally crochet/knit/darn this! (in $\mathbb{R}^3$!)



then identify the red circles...

ex. (maybe) Challenge/review for 215a:

Show that the dunce cap is $\simeq D^2 \simeq *$ (homotopy equiv to a point!).

Soft intro to simplicial homology chain complex

We wish to translate the Δ-cpx structure to algebra, in the form of a chain complex of abelian groups.

↪ later we'll see that the homology of this chain cpx does not depend on the choice of Δ-cpx structure. But for now we just want to build a chain cpx.

Chain complex: 2 pieces of data:

- a $\mathbb{Z}$-mod C_i for each dimension $i \in \mathbb{Z}$ "chain groups" (most are 0, and for us $C_i = 0$ when $i < 0$).
- a "differential", ie maps $\partial_i : C_i \longrightarrow C_{i-1}$ $\forall i$. such that $\partial^2 = 0$ ie $\partial_{i+1} \circ \partial_i = 0$ $\forall i$.

In our case, $C_i = \mathbb{Z}\langle \{\sigma_\alpha\} \mid n(\alpha) = i \}$

and ∂_i records the image of $\partial \Delta^{n(\alpha)}$.

↑ also sometimes called "the boundary map" for this reason.

eg. $\Delta^1 = \underset{v_0}{\bullet}\longrightarrow\underset{v_1}{\bullet}$ is a space with a Δ cpx structure, where the σ_α are inclusions.

0-simplices: $\sigma_\alpha : \Delta^0 \longrightarrow \Delta^1$, $v_0 \longmapsto v_0$ $\qquad$ $\sigma_\beta : \Delta^0 \longrightarrow \Delta^1$, $v_0 \longmapsto v_1$

1-simplices: $\sigma_\gamma : \Delta^1 \longrightarrow \Delta^1 = id_{\Delta^1}$.

Blue Δ^1 is thought of as the top space, ie $X = \Delta^1$.

<u>Chain complex</u> let $X = \Delta^1$

Here's whats actually happening:

chaingroups: $C = \bigoplus_{i\in\mathbb{Z}} C_i$, $C_0 = \mathbb{Z}\langle\sigma_\alpha, \sigma_\beta\rangle$, $C_1 = \mathbb{Z}\langle\sigma_\gamma\rangle$,

all other $C_i = 0$.

differential: $\partial_1 : C_1 \longrightarrow C_0$ is determined by

$$\sigma_\gamma \longmapsto \sigma_\beta - \sigma_\alpha$$

(all other $\partial_i = 0$)

How did I get this? We actually think about where the boundaries of $\Delta^{n(\alpha)}$ are sent!

We actually may identify σ_α with its image <u>in this example</u>

So we may write (as we did Wednesday):

$$\partial[v_0, v_1] = \underset{\text{final}}{[v_1]} - \underset{\text{initial.}}{[v_0]}$$

↑ I omitted the subscript b/c I can think of $\partial = \bigoplus_i \partial_i$ as a <u>total</u> differential on the module $C = \bigoplus C_i$

Since all other $\partial_i = 0$, it is clear that ∂ satisfies $\partial^2 = 0$.

↳ <u>ex.</u> If you've never seen this before, write $\partial = \partial_0 + \partial_1 + \dots$ and write down the <u>graded</u> pieces of the equation to deduce that $\partial_{i+1} \circ \partial_i = 0$.

Quick start guide to chain complexes, homology

Let's start with some algebra. Then we'll describe simplicial homology formally with additional standard notation, terminology

- Chain groups: $C = \bigoplus C_i$
- differential $\partial = \sum \partial_i$, where $\partial^2 = 0$
- chain complex: (C, ∂)
- homology functor: $\{\text{chain cpxs}\} \longrightarrow \{\text{graded } \mathbb{Z}\text{-mods}\}$

by $$H_i = \frac{\overset{\text{cycles}}{\ker \partial_i}}{\underset{\text{boundaries}}{\operatorname{im} \partial_{i+1}}}$$

note This is only ok to write if you know $\operatorname{im}\partial_{i+1} \subset \ker \partial_i$, This is ensured by $\partial_{i+1} \circ \partial_i = 0$.

- Two cycles are homologous if they differ by a boundary. Then they represent the same homology class.

- The point of homology: associating different decompositions of the space X (diff Δ-structures) may yield different chain complexes, but these all compute the same homology groups $\{H_i(X)\}_{i \in \mathbb{Z}}$. (all together, we write $H_*(X)$).

  * This point is very top-spaces-centric. In general:

  $$\underbrace{\text{chain htpy equivalence}}_{\substack{\text{stronger equivalence; analogous to} \\ \text{"homotopy equiv" for top spaces}}} \Longrightarrow \underbrace{\text{quasi-isomorphism}}_{:= \substack{\text{have the same} \\ \text{homology groups}}}$$

- "graded" just means you associate integers (the dimension) to different pieces of (C, ∂)

Simplicial Homology

- let X be a Δ-complex.
- let $\Delta_n(X)$ be the free abelian group generated by the open n-simplicies e_α^n of X.
  - By Δ-1, it's ok to think of these e_α^n as part of X, the target of the map $\sigma_\alpha : \Delta^n \to X$

    We call σ_α the characteristic map of e_α^n.

  - We write each element as finite formal sums

$$\sum_{\{\alpha \in A \mid n(\alpha) = n\}} c_\alpha \cdot \sigma_\alpha$$

↑ b/c we are working with $\mathbb{Z}\sigma_\alpha \oplus \mathbb{Z}\sigma_\beta \oplus \dots$ ← direct sums.

or we could write $\sum c_\alpha e_\alpha^n$, but I won't, in general...

- The boundary homomorphism $\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$ is given by

basis elt.

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \underbrace{\sigma_\alpha \Big|_{[v_0, \dots, \hat{v_i}, \dots, v_n]}}_{\text{by } \Delta\text{-2 we have this map as a basis element for } \Delta_{n-1}(X)}$$

↑ where $n(\alpha) = n$

Claim (lemma 2.1) Let $\mathscr{C} = \bigoplus_n \Delta_n(X)$. and $\partial = \sum_n \partial_n$

Then $(\mathscr{C}, \partial)$ is a chain complex.

Pf.

It suffices to show that the composition

$$\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$$

is the zero map.

Calculate directly from the definition:

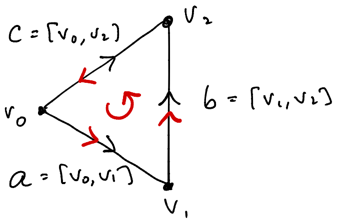
$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v_i}, \dots, v_n]}$$

$$\left.\begin{aligned} \partial_{n-1}\partial_n(\sigma) = & \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \hat{v_j}, \dots, \hat{v_i}, \dots, v_n]} \\ & + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \hat{v_i}, \dots, \hat{v_j}, \dots, v_n]} \end{aligned}\right\} \text{opp signs. These cancel.}$$

note the parity change b/c v_i was deleted!

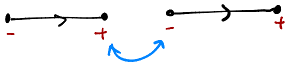
■

Rmk What's happening topologically?



on a simplex, with orientations of codim 1 faces determined by the ∂ orientation of the interior (ie "cyclic", at least in the shown dimension),

all codim 2 faces are used twice, but w/ opp. sign (orientation)



glue only + to -
never - to - or + to +!

Defn We call $H_*(\mathscr{C}, \partial)$ the simplicial homology of X

and write it as $H^\Delta(X) = \bigoplus_n H_n(X)$