Lecture 13

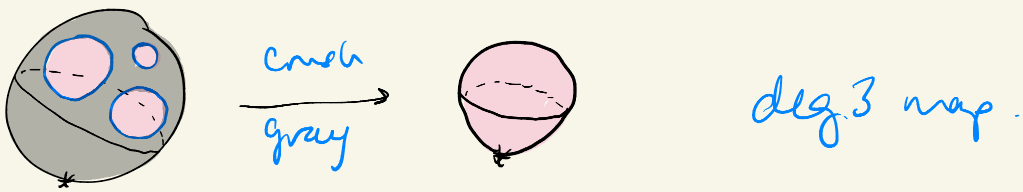
Cell complexes: built from cells $e^n \cong D^n$.

Attaching maps e^n onto X^{n-1} given by $\partial e^n \cong S^{n-1} \longrightarrow S^{n-1}$

Given $f: S^n \longrightarrow S^n$, induced $f_*: \tilde{H}_n(S^n) \longrightarrow \tilde{H}_n(S^n)$ is a map $\mathbb{Z} \longrightarrow \mathbb{Z}$, $\alpha \longmapsto d\alpha$

call d the <u>degree of f</u>, denoted <u>$\deg f$</u>.

Very useful concept. Intuition for degree:



<u>useful properties</u>

(a) $\deg \mathbb{1} = 1$

(b) $\deg f = 0$ if f not surjective.

Recall Cell complex built from cells $e^k \cong \mathbb{D}^k$ but attaching inductively the e^k_j to X^{k-1} the k-1 skeleton.

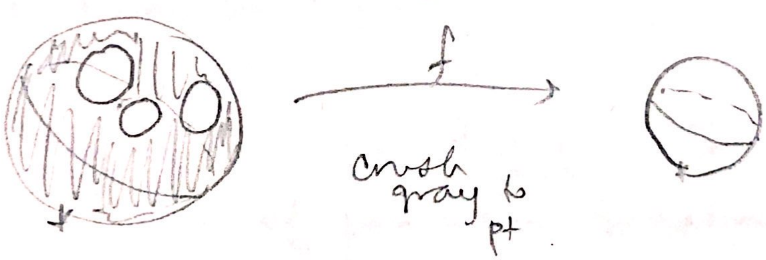
attaching maps : $\partial \mathbb{D}^k = S^{k-1} \longrightarrow S^{k-1} \subset X^{k-1}$

Observe $f: S^n \to S^n$ $n > 0$ then on top homo. $f_*: H_n(S^n) \longrightarrow H_n(S^n)$

$$\mathbb{Z} \longrightarrow \mathbb{Z}$$
$$\alpha \longmapsto d\alpha$$

is characterized by d, ~~the~~ called the degree of f, written deg f.

Picture Very important concept, here's intuition on degree:



degree 3

List of properties ("basic") (pf below)

(a) $\deg \mathbb{1} = 1$

$\mathbb{1}_* = \mathbb{1}. = \text{'}\times 1\text{'}$

(b) f not surjective $\Rightarrow$ $\deg f = 0.$

$\hookrightarrow$ $\Rightarrow$ choose $x_0 \in S^n - f(S^n)$

$$\underset{\text{dom}}{S^n} \xrightarrow{f'} \underset{\text{tar}}{S^n - \{x_0\}} \hookrightarrow S^n$$

$$H_n(\underbrace{S^n - \{x_0\}}_{\text{contractible!}}) = 0.$$

(c) $f \simeq g \Rightarrow \deg f = \deg g$ since $f_* = g_*$.

* converse is harder! will be Corollary 4.25 (around spheres nice?)

☆ come back to.

(d) $\deg fg = \deg f \deg g$ since $(fg)_* = f_* g_*$

$\Rightarrow$ if you have grouping deg fn is homo. $d : G \to \mathbb{Z}$.

Consequently, ~~deg f~~

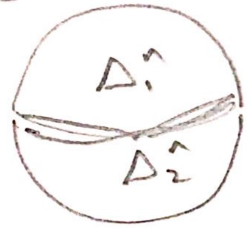
- f is hom. equi $\Rightarrow$ $\deg f = \pm 1$ (~~[illegible]~~)

$$\left(fg \simeq \mathbb{1} \Rightarrow \deg f \deg g = \deg \mathbb{1} = 1 \right)$$

(e) f is reflection of S^n that fixes an equator S^{n-1} and exchanges two hemispheres

$\Rightarrow$ deg. $f = -1$.

$\hookrightarrow$ check by literally using the Δ-cpx structure



to see $\Delta_1^n - \Delta_2^n \longrightarrow \Delta_2^n - \Delta_1^n$.

(f) on the other hand, the antipodal map

$-\mathbb{1} : S^n \longrightarrow S^n$ has degree $(-1)^{n+1}$!

$x \longmapsto -x$

b/c it is the product of $n+1$ reflections?

$p \in S^n$ is $(x_0, \ldots, x_n) \mid \sqrt{x_0^2 + \cdots + x_n^2} = 1$

$-\mathbb{1}(x_0, \ldots, x_n) = (-x_0, \ldots, x_n)$

Observe $-\mathbb{1}$ has no fixed points.

In fact . . .

(8) **Fact**
maps $S^n \to S^n$ w/o fixed points are $\simeq -\mathbb{1}$!

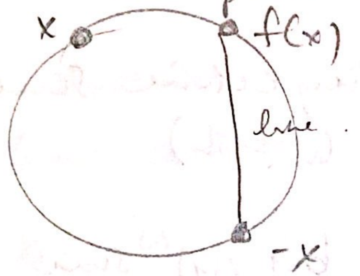
Pf.

If $f(x) \neq x$ then there is a line

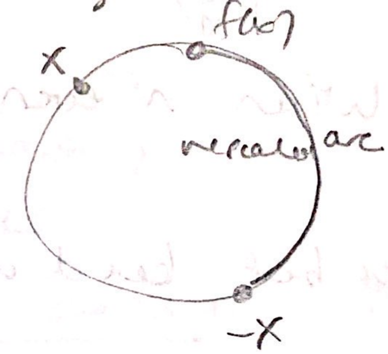
$f(x) \to -x$: $t \mapsto (1-t) f(x) - tx$

$t \in [0,1]$

does not pass through the origin



vertical



Then

$$f_t(x) = \frac{(1-t) f(x) - tx}{|(1-t) f(x) - tx|}$$ clearly continuous.

is a htpy from f to $-\mathbb{1}$.

(* no antipodal mapping ⇒ htpc to identity?)

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prop. 2.24 / Cor $\mathbb{Z}/2$ is the only nontrivial gp that can act freely on S^n if $\underline{n \text{ is even}}$

(prop. 2.29).

— discuss free action?

- homeos have degree ± 1
- $d : G \to \{\pm 1\}$.
- free ⇒ $d(g) = (-1)^{n+1}$ when $g \neq 1$

~~degree of $G \to \{\pm 1\}$, $g \neq 1$ ⇒ deg =~~

pf

- homos have degree ± 1 (~~...~~ invertible)

- $G \curvearrowright S^n \rightsquigarrow \deg$ for $d: G \to \{\pm 1\}$.

  (homomorphism)

  b/c ~~fs~~ functoriality $\deg fg = \deg f \deg g$.

- action free $\Rightarrow d(g \neq 1) = (-1)^{n+1}$ by (g).

- When n even, d is surjective since $(-1)^{n+1} = -1$. (if $G \neq 1$)

- ~~...~~ but $\ker d$ is trivial by (g) since action is free

$\Rightarrow G \leq \mathbb{Z}/2. \Rightarrow G = \mathbb{Z}/2$

Q

Name: ____________ SID: ____________

Mat 21A Final Exam

Below you will find 8 problems for a total of 80 points (and one extra credit problem). You must show all your work to receive full credit, just the answer (correct or not) is not enough. Calculators, books, and notes are not allowed.

You got this! Good luck!

Problem 1 (10 points) : Suppose that $f(x)$ is continuous and differentiable on the interval $[-1, 2]$ such that $f(-1) = 1$ and $f'(x) \geq 2$. What is the smallest value that $f(2)$ can be?

Fun application of degree (Hairy ball thm?)

thm 2.28 S^n has a cts field of nonzero tangent vector iff n is odd.

[i.e. TS^n has a section iff n is odd
see VBs.]

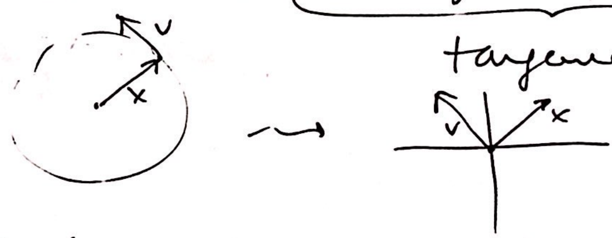
pf.

[Interesting direction: ~~nodd~~ $\Rightarrow$
TS^n has section $\Rightarrow$ n is odd]

$x \mapsto v(x)$ is the tangent v. field.

$\Leftrightarrow$ $x \perp v(x)$ are orthogonal as vectors in $\mathbb{R}^{n+1}$

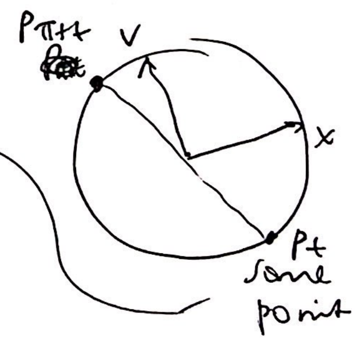
tangency



All $v(x) \neq 0$ so normalize so that wlog we assume $|v| = 1$.

now, consider the vectors

$$p_t = (\cos t)x + (\sin t)v(x) \in S^n,$$



so by letting $t \in [0, \pi]$ get $p_t \rightsquigarrow -p$.

All together this is a cts map.

$$f_t(x) = \cos(t)(x) + (\sin t)\,v(x) : \mathrm{id}_{S^n} \to -\mathbb{I}_{S^n}$$

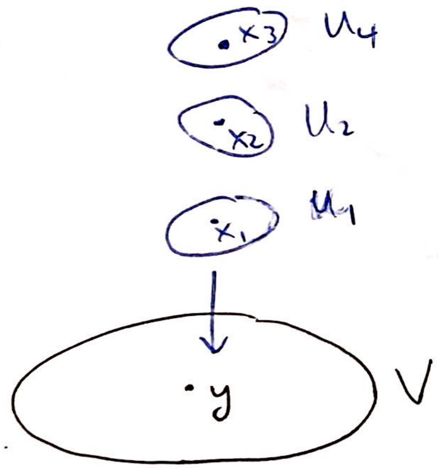
local degree + how to compute degree

Suppose $f: S^n_{src} \longrightarrow S^n_{tar}$, $n > 0$

w/ property that $\boxed{\text{if } y \in S^n_{tar},\ f^{-1}(y) = \text{finitely many points } x_1, \ldots, x_m}$

PANCAKES on plate

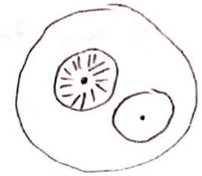
Let U_i be disjoint nbhds of x_i mapping into a nbhd V of y:

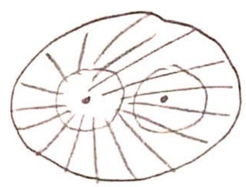


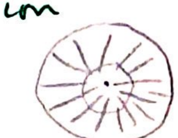
Then $f(U_i - x_i) \subset V - y$

and we get the following commutative diagram...

(~~defn.~~ of local degree)







$$
\begin{tikzcd}[column sep=large, row sep=large]
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\text{excision}", "\cong"'] \arrow[d, "k_i\ \text{inclusion}"] \arrow[dr, phantom, "\substack{\bullet\ \deg f|_{x_i} \\ \text{``local degrees''} \\ \circlearrowright\ \text{naturality}}"] \arrow[dl, phantom, "\circlearrowright\ \text{composition}", shift left=6] & H_n(V, V-y) \arrow[d, "\text{excision}", "\cong"'] \\
H_n(S^n_{src}, \underset{\text{contractible}}{S^n - x_i}) & H_n(S^n_{src}, S^n - f^{-1}(y)) \arrow[l, "p_i"', "\substack{\cancel{\text{inclusion}} \\ \text{projection} \\ \text{from } \oplus}"] \arrow[r, "f_*"] \arrow[dr, phantom, "\circlearrowright\ \text{naturality}"] & H_n(S^n_{tar}, \underset{\text{contractible}}{S^n - y}) \\
 & H_n(S^n_{src}) \arrow[ul, "\text{LES}", "\cong"'] \arrow[u, "j", "\substack{\text{LES} \\ \cancel{\text{LES}} \\ \cancel{\text{inclusion}}}"'] \arrow[r, "f_*"] \arrow[ul, phantom, "\circlearrowright\ \text{composition}", shift right=6] & H_n(S^n_{tar}) \arrow[u, "\cong", "\text{LES}"']
\end{tikzcd}
$$

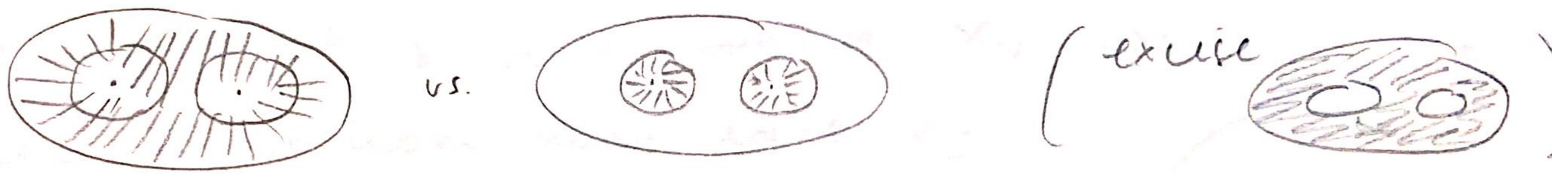
$\mathbb{Z}$ $\qquad\qquad$ $\mathbb{Z}$

use green arrows to identify top two groups w/ $\mathbb{Z}$.
Obtain local degree for top horiz arrow.

<u>prop. 2.30</u> (With the assumptions above) $\deg f = \sum_i \deg f|_{x_i}$.

Pf.

- By excision, central term $H_n(S^n, S^n - f^{-1}(y)) = \bigoplus H_n(U_i, U_i - x_i)$



each $H_n(U_i, U_i - x_i) \cong \mathbb{Z}$, k_i is the inclusion map for the ith summand.

- p_i are the projections from the direct sum. (see top Δ)
- bottom ∇ commutes $\rightsquigarrow$ $p_i \circ j(1) = 1$

$\rightarrow$ $j(1) = (1, \dots, 1) = \sum_i k_i(1)$

- upper square $\square$ commutes $\Rightarrow$ middle f_* sends $k_*(1) \mapsto \deg f|_{x_i}$

$\Rightarrow$ $\sum_i k_i(1) = j(1) \longmapsto \sum_i \deg f|_{x_i}$

- lower square commutes: really identifies $\deg f$ with $\sum_i \deg f|_{x_i}$

$\blacksquare$

example application

eg. 2.32. View $S^1 \subset \mathbb{C}$ as unit circle and consider $f: S^1 \to S^1$, $z \mapsto z^k$

say $k>0$ ($k=0$ easy, $k<0$ can be obtained from $k>0$).

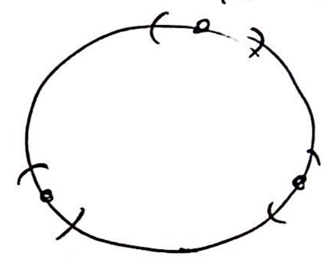
- $f^{-1}(y)$ consists of k points ~~where~~ $x_1, \ldots, x_k$.

  f is a local hom near each x_i

- homotope $f|_{U_i}$ (near x_i),

  stretch by factor of k (reparam)

  so that we can view $f|_{U_i}$ ~~as~~ as a restriction of a rotation (by θ_i).



  * rotations are homeos, and thus is orientation preserving $\Rightarrow$ $\deg f|_{x_i} = +1$.

$\Rightarrow$ $\deg f = k$.

rmk. can repeatedly suspend this f to get deg k maps $S^n \longrightarrow S^n$ of degree k.

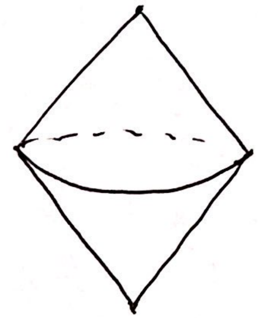
prop. 2.33. Suspension preserves degree..

Aside Suspension of maps

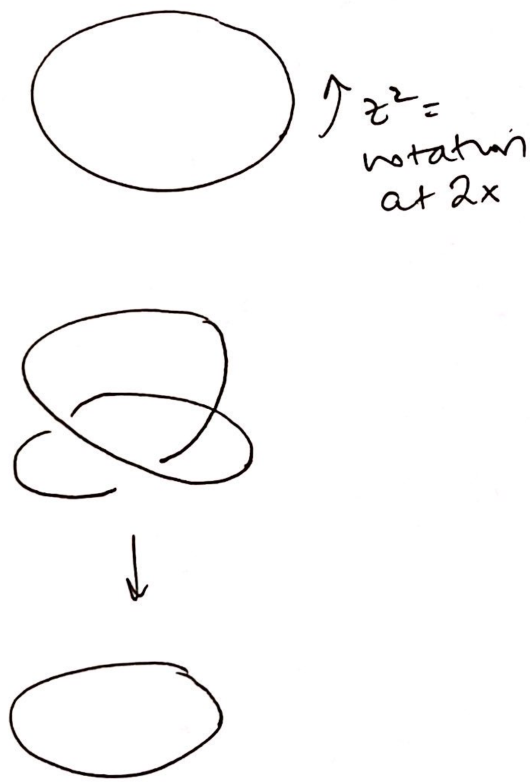
$f: X \to Y$ cts map

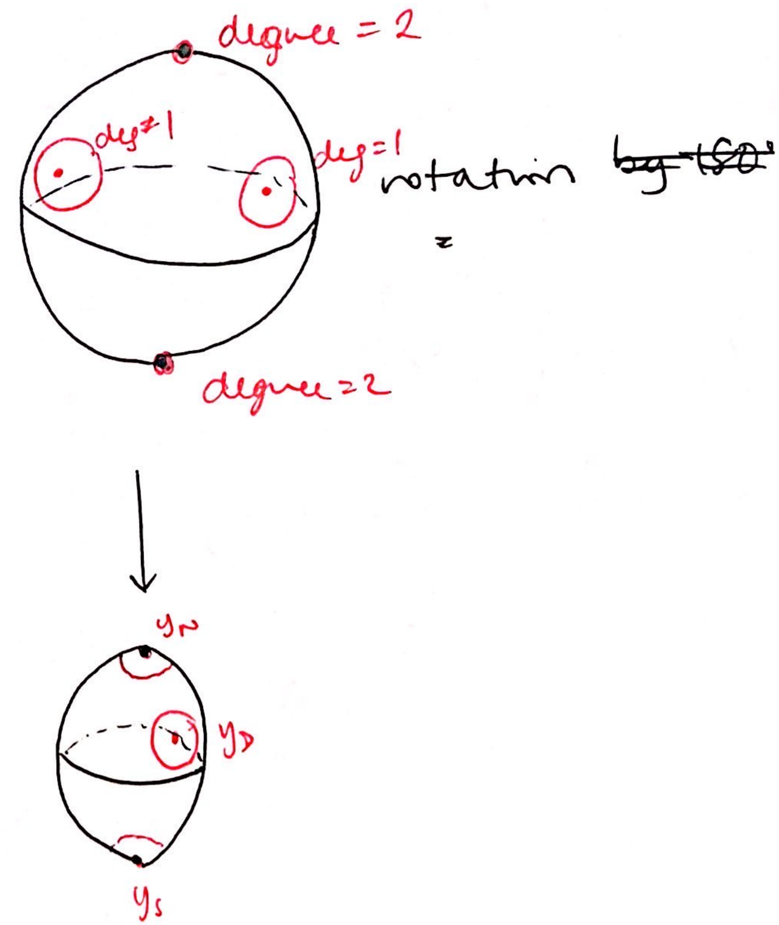
$Sf: SX \to SY$

$[x, t] \mapsto [f(x), t]$



eg.





A few facts about singular Hom. of CW cpxs!

<u>Lemma 2.34</u> If X is a CW cpx, then

(a) Say X^n is obtained by attaching $\{e_\alpha^n\}_{\alpha\in A}$ to X^{n-1}.

$$H_k(X^n, X^{n-1}) \cong \begin{cases} 0 & \text{for } k \neq n \\ \bigoplus_\alpha \mathbb{Z}\langle e_\alpha^n\rangle \end{cases}$$

(X^n, X^{n-1}) is a good pair, and $X^n/X^{n-1} \approx \bigvee_\alpha S_\alpha^n$ (this is the proof)

(b) $H_k(X^n) = 0$ for $k > n$.

↳ If X is finite dim then $H_k(X) = 0$ when $k > \dim X$.

no interesting homology when there are no singular chains either (not the proof just a mnemonic)

(c) The inclusion $i: X^n \hookrightarrow X$ induces an isom $i_*: H_k(X^n) \longrightarrow H_k(X)$ if $k < n$.

kth homology only depends on the X^{k+1} skeleton again makes sense thinking about simplicial hom.

pf. (a) done

Consider LES for pair (X^n, X^{n-1}):

$$\underbrace{H_{k+1}(X^n, X^{n-1})}_{\substack{=0 \\ \text{if } k\neq n-1}} \longrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow \underbrace{H_k(X^n, X^{n-1})}_{\substack{=0 \\ \text{if } k=n}}$$

- if $k > n$ then we have
$$H_k(X^n) \cong H_k(X^{n-1}) \cong \dots \cong H_k(X^0) = 0.$$
This proves (b).
- if $k < n$, then
$$H_k(X^n) \cong H_k(X^{n+1}) \cong \dots \cong H_k(X^{n+m}).$$
↳ so if $X = X^d$, then this proves (c)

pf of (c) for infinite dim'l case (sketch/discussion)

- again need to use the fact that singular chains have cpt image and thus meets only finitely many cells (last time it was simplices for H^Δ)
- so each chain lives in a finite skeleton X^m
- ...