Lecture 16

Recall Lemma 2.34 Let X be a cell complex.

(a) $H_n(X^n, X^{n-1}) \cong \mathbb{Z}\langle \{n \text{ cells of } X\}\rangle$ wedge of spheres

o/w $H_k(X^n, X^{n-1}) = 0$ when $k \neq n$.

(b) When $k > n$, $H_k(X^n) = 0$ no k-cells to say anything about

(c) The map $H_k(X^n) \longrightarrow H_k(X)$

induced by inclusion $X^n \hookrightarrow X$

- is $\cong$ when $k < n$ X^n knows all when $k<n$
- is surjective when $k=n$ need X^{n+1} to know relations

The cellular chain cpx $C_\bullet^{cw}(X)$: (defn)

ref. $0 \longrightarrow C_k(X^{n-1}) \xrightarrow{i_k} C_k(X^n) \xrightarrow{j_k} C(X^n, X^{n-1}) \longrightarrow 0.$ (Hatcher notation)

$\partial[\alpha] = [\partial\alpha]$

$$\begin{tikzcd}
 & & H_n(X^n) \arrow[dr, "j_{*,n}"] & & & & \\
{} \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & {} \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{*,n-1}"'] & &
\end{tikzcd}$$

Clear: $d^2 = 0$ b/c $d^2 = j_* \circ \partial \circ j_* \circ \partial$ but $\partial \circ j_*$ is composition of two consecutive maps in a LES, which is in particular a chain cpx.

Why does this compute singular homology?

__thm__ $H_\bullet^{CW}(X) \cong H_\bullet(X)$ (for cell complex X)

$$\begin{tikzcd}
 & & & & H_n(X^{n+1}, X^n) \underset{(a)}{=} 0 & & \\
 & 0 \underset{(b)}{=} H_n(X^{n-1}) \arrow[dr] & & H_n(X^{n+1}) \underset{(c)}{\cong} H_n(X) \cong H_n(X^n)/\ker i_{*,n} = H_n(X^n)/\operatorname{im}\partial_{n+1} \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur, "i_{*,n}"] \arrow[dr, "j_{*,n}"] & & & & \\
 {} \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[dr, "\partial_n"] \arrow[rr, "d_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & {} \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{*,n-1}"] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

① $j_{*,n}$ injective $\Rightarrow$ takes $\operatorname{im}\partial_{n+1}$ $\cong$ly onto $\operatorname{img}(d_{n+1})$

$\Rightarrow \operatorname{im}\partial_{n+1} \cong \operatorname{im} d_{n+1}$

② $j_{*,n}$ takes $H_n(X^n)$ $\cong$ly onto $\ker\partial_n \subset H_n(X^n, X^{n-1})$.

$j_{*,n-1}$ injective $\Rightarrow \ker\partial_n \cong \ker d_n$.

$\Rightarrow H_n(X^n) \cong \ker d_n$

$$\Rightarrow H_n(X) \cong H_n(X^n)/\operatorname{im}\partial_{n+1} \cong \ker d_n/\operatorname{im} d_{n+1} \qquad \square$$

Proof: From the diagram above, $H_n(X)$ can be identified with $H_n(X^n)/\operatorname{Im}\partial_{n+1}$. Since j_n is injective, it maps $\operatorname{Im}\partial_{n+1}$ isomorphically onto $\operatorname{Im}(j_n\partial_{n+1}) = \operatorname{Im} d_{n+1}$ and $H_n(X^n)$ isomorphically onto $\operatorname{Im} j_n = \operatorname{Ker}\partial_n$. Since j_{n-1} is injective, $\operatorname{Ker}\partial_n = \operatorname{Ker} d_n$. Thus j_n induces an isomorphism of the quotient $H_n(X^n)/\operatorname{Im}\partial_{n+1}$ onto $\operatorname{Ker} d_n/\operatorname{Im} d_{n+1}$. $\square$

How to actually compute d_n?

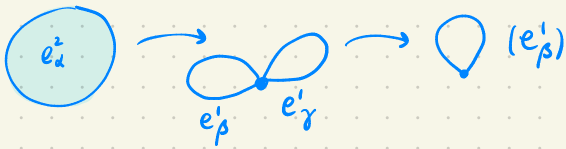
Cellular Boundary Formula. $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$ *where* $d_{\alpha\beta}$ *is the degree of the map* $S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$ *that is the composition of the attaching map of* e^n_α *with the quotient map collapsing* $X^{n-1} - e^{n-1}_\beta$ *to a point.*

(pf will basically be omitted)

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$$

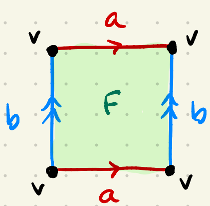
degree of map

$$S^{n-1}_\alpha \longrightarrow X^{n-1} \longrightarrow S^{n-1}_\beta$$



<u>eg.</u> Torus

$X =$



$d_{Fa} = 1 - 1 = 0$

$d_{Fb} = 1 - 1 = 0$

Whole calculation: Actually $d = 0$:

$$0 \longrightarrow \mathbb{Z}\langle F\rangle \longrightarrow \mathbb{Z}\langle a,b\rangle \longrightarrow \underline{\mathbb{Z}\langle v\rangle} \longrightarrow 0$$

$$F \longmapsto 0 \qquad a \longmapsto 0$$
$$b \longmapsto 0$$

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^2 & n = 1 \\ 0 & \text{o/w} \end{cases}$$

<u>Q.</u> Another space with same homology? $\rightsquigarrow$ Moore spaces

In fact, by similar calculation we get

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z}^{2g} & n=1 \\ 0 & \text{o/w.} \end{cases}$$

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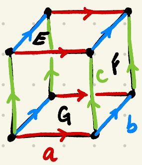
eg. $T^3 = S^1 \times S^1 \times S^1$



- all vertices are v
- all parallel edges are the same edge: a, b, c
- all parallel faces are the same face E, F, G

  $\partial E = aba^{-1}b^{-1}$, $\partial F = bcb^{-1}c^{-1}$, $\partial E = aca^{-1}c^{-1}$ ← topological statement.

  $\Rightarrow d_2(E), d_2(F), d_2(G)$ are all 0!
- Single 3-cell filling in the cube.

  $d(e^3) = E - E + F - F + G - G$

$$H_n(T^3) \cong \begin{cases} \mathbb{Z} & n=0,3 \\ \mathbb{Z}^3 & n=1,2 \\ 0 & \text{o/w} \end{cases}$$

(notice a pattern?) $\rightsquigarrow$ Künneth formula

<u>HW</u>: $\mathbb{RP}^n$, using what you know about antipodal maps + degree from last week (more explicitly than in Hatcher)

Discussion 6

Good example to have in pocket In odd dimensions: lens spaces

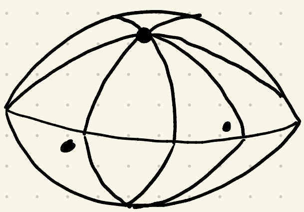
(3D) Let p, q be coprime integers. We may assume $p > 0$.

Consider S^3 = unit sphere in $\mathbb{C}^2$.

Then
$$(z_1, z_2) \xmapsto{\tau} (e^{2\pi i/p} z_1, e^{2\pi i q/p} \cdot z_2)$$
is a generator of a free $\mathbb{Z}/p\mathbb{Z}$ rotation action.

defn. The lens space $L(p,q)$ is S^3/τ.

We can build $L(p,q)$ as follows:

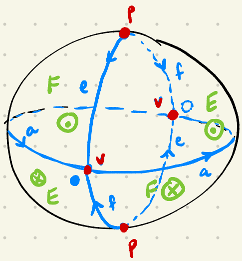


Cut the top and bottom hemispheres into p wedges. Glue the Δ shown in the S hemisphere to the Δ on the N hemisphere that is a q/p turn away.

<u>eg.</u> (More complicated cell structure than necessary)

<u>$L(2,1) = \mathbb{RP}^3$</u>: Check that this gives the antipodal map.

Build a CW complex:



$d_0 = 0$

$d_1(a) = 0 \quad d_1(e) = v - p \quad d_1(f) = v - p$

$d(E) = a - f + e \quad d(F) = a - e + f$

$d(e^3) = 0.$

$\ker d_1 = \mathbb{Z}\langle a, e-f \rangle \qquad \operatorname{im} d_2 = \mathbb{Z}\langle a-f+e, a-e+f \rangle$

$= \mathbb{Z}\langle a, a+\cancel{e-f} \rangle \qquad = \mathbb{Z}\langle 2a, a+\cancel{e-f} \rangle$

$H_1 \cong \mathbb{Z}/2\mathbb{Z}$

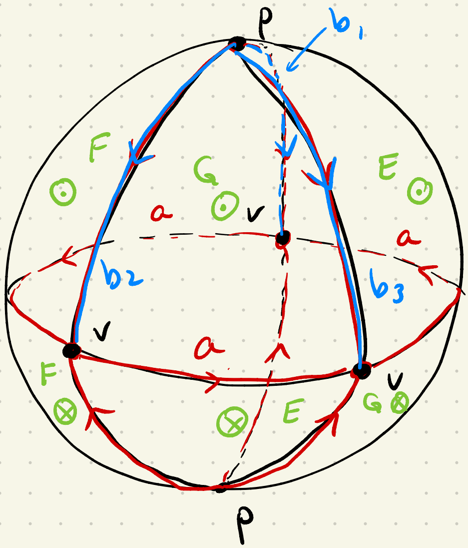
$\ker d_2 = 0, \ \operatorname{im} = 0 \implies H_1 \cong 0$

$H_3 \cong \mathbb{Z}.$

In general?

$$H_n(L(p,q)) = \begin{cases} \mathbb{Z} & n = 0,3 \\ \mathbb{Z}/p\mathbb{Z} & n = 1 \\ 0 & \text{o/w} \end{cases}$$

* However, $L(5,2) \not\cong L(5,1)$! (Alexander 1919, htpy type diff)



Compute H_1:

$d_1(a) = 0$

$d_1(b_i) = v - p$

$d_2(E) = a - b_1 + b_3$

$d_2(F) = a - b_2 + b_1$

$d_2(G) = a - b_3 + b_2$

$\ker d_1 = \langle a, b_1 - b_2, b_2 - b_3, \cancel{b_3 - b_2} \rangle$

$= \langle a, a \cancel{- b_2 + b_1}, a \cancel{- b_2 + b_3} \rangle$

$\operatorname{im} d_2 = \langle a - b_1 + b_3, a - b_2 + b_1, a - b_3 + b_2 \rangle$

$\langle a \cancel{- b_1} + b_3, a \cancel{- b_2} + b_1, 3a \rangle$

Euler characteristic (shadow of homology)

Many different interpretations (eg. w/ vector fields)

for us:

defn For a CW cpx X, $\chi(X) = \#\,$0-cells $- \#\,$1-cells $+ \#\,$2-cells $- \dots$

prop. $\chi(X)$ doesn't depend on the CW complex structure.

In fact, $\chi(X) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rk} H_n(X)$

This is a purely algebraic fact + works for all chain cpxs of abelian groups. ($\mathbb{Z}$-modules)

lemma. If $0 \to A \to B \to C \to 0$ is a SES of finitely generated $\mathbb{Z}$-mods then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

$C \cong B/A$, $\operatorname{rank} C = \operatorname{rank} B - \operatorname{rank} A$.

thm. $\chi(\mathcal{C}_\bullet) = \chi(H_\bullet(\mathcal{C}))$ $(\mathcal{C}, d)$ a chain cpx.

pf.

Z_n = cycles in C_n, B_n = bdys in C_n, $H_n = Z_n / B_n$

① $0 \to B_n \to Z_n \to H_n \to 0$

$\Rightarrow \operatorname{rk} Z_n = \operatorname{rk} B_n + \operatorname{rk} H_n$

How to relate to C_n? C_n surjects onto its image under d.

② $0 \to \underbrace{Z_n}_{\ker d_n} \to C_n \xrightarrow{d_n} \underbrace{B_{n-1}}_{\operatorname{im} d_n} \to 0$

$\Rightarrow \operatorname{rk} C_n = \operatorname{rk} B_{n-1} + \operatorname{rk} Z_n$

$\Rightarrow \operatorname{rk} C_n = \operatorname{rk} H_n + \operatorname{rk} B_n + \operatorname{rk} B_{n-1}$

In alt sum, the $\operatorname{rk} B_n + \operatorname{rk} B_{n-1}$ cancels out. //

eg. $\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g$

eg. $\chi(N_\gamma) = 1 - \gamma + 1 = 2 - \gamma$

nonorientable

Moore Spaces

<u>defn.</u> A Moore space $M(G,n)$ is a space
G = abelian gp, $n \geq 1$

$H_n(M(G,n)) = G$ and $\tilde{H}_i(M(G,n)) = 0 \ \forall \ i \neq n$,
and if $n \geq 2$, is simply connected.

eg. • $M(\mathbb{Z},n)$ is easy to build
• $M(\mathbb{Z}/k\mathbb{Z}, n)$ via degree k map of sphere
• In general need presentation of G to construct

eg. $\mathbb{RP}^2$ is an $M(\mathbb{Z}/2\mathbb{Z}, 1)$

<u>Rmk</u> If you want a space level construction to be interesting (eg. have interesting cohom ring), don't want (wedge of) Moore spaces because they basically give only the data of the homology groups.

<u>Compare</u> An Eilenberg-Maclane space $K(\pi, n)$ has $\pi_n(X) = \pi$.
otherwise $\pi_k(X) = 0$

These (in my opinion) are much harder to think about.

eg. $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$
b/c S^∞ is contractible.
$\hookrightarrow$ group cohomology

Mayer-Vietoris Sequences

Say $A, B \subset X$ s.t. $X = \mathring{A} \cup \mathring{B}$. There is a LES

$$\cdots \to H_n(A\cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A\cap B) \to \cdots\cdots \to H_0(X) = 0.$$

$= C^{\mathcal{U}}$ for the cover $\{A, B\}$

- $C_n(A+B) \subset C_n(X)$ (cf. $C_n(A) \oplus C_n(B)$)
- ∂ takes $C_n(A+B) \circlearrowleft \Rightarrow C_\bullet(A+B)$ sub complex
- The inclusions $C_\bullet(A+B) \hookrightarrow C_\bullet(X)$ induces $\cong$ on homo.

SES

$$0 \to C_n(A\cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$$

$$x \longmapsto (x, -x)$$

$$(x, y) \longmapsto x+y$$

So "φ_*" is Φ in the LES notation, $\Psi = $ "ψ_*".
except that we identify $H_0(A+B)$ with $H_0(X)$

What is conn. map ∂? $\partial : H_n(X) \longrightarrow H_{n-1}(A\cap B)$
$[z]$

Cycle $z \xrightarrow{\text{bary. subdiv}} z = x+y$ where $x \in C_n(A)$, $y \in C_n(B)$

(* just chains - not nec'ly cycles)

$\partial z = 0 \Rightarrow \partial(x+y) = 0 \Rightarrow \partial x = -\partial y.$

Now look back at defn of ∂ by snake:

$$\begin{array}{ccc} & & C_{n-1}(A\cap B) \\ & & \downarrow \\ C_n(A)\oplus C_n(B) & \to & C_{n-1}(A) \oplus C_{n-1}(B) \\ \downarrow & & \\ C_n(A+B) & & \end{array}$$

$z = x+y \longleftarrow (x, y) \longrightarrow (\partial x, -\partial y) \longleftarrow \partial x.$

$\Rightarrow \partial[z] = \partial x \ (= -\partial y).$

Rmks

① There is analogous M-V seq^n for reduced, where we augment by

$$\begin{array}{ccccccccc} 0 \to & C_0(A\cap B) & \xrightarrow{\varphi} & C_0(A)\oplus C_0(B) & \xrightarrow{\psi} & C_0(A+B) & \to 0 \\ & \downarrow \varepsilon & & \downarrow \varepsilon\oplus\varepsilon & & \downarrow \varepsilon & \\ 0 \to & \mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow 0 \\ & a & \longmapsto & (a,-a) & & & \\ & & & (b,c) & \longmapsto & b+c. & \end{array}$$

② We'll see that $H_1(X) = \text{abelianization}(\pi_1(X))$ for X path-cntd (here no x_0 in notation)

Consider reduced M-V seq^n. Then H_0's are all 0 since everything is path cntd.

$\Rightarrow$ $H_1(X) \cong H_1(A)\oplus H_1(B) \big/ \ker \Psi = \operatorname{Im}\Phi$

which is abelianized statement of Seifert Van-Kampen.

③ Useful in induction arguments:

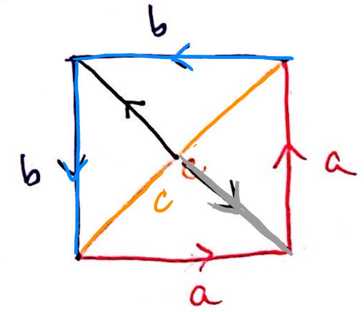
eg. Let $X = S^n$, A = N-hemisphere, B = S-hemisphere.

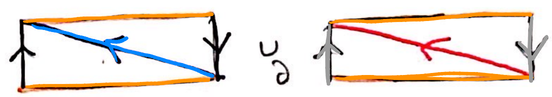
$\Rightarrow A\cap B = S^{n-1}$ (path cntd if $n\geq 2$.)

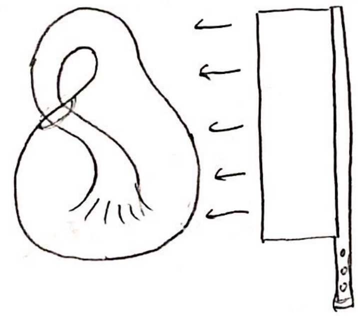
$A, B \cong D^n \Rightarrow \widetilde{H}_i(A)\oplus \widetilde{H}_i(B)$ are all 0.

$\Rightarrow$ isomorphisms $\widetilde{H}_i(S^n) \approx \widetilde{H}_{i-1}(S^{n-1})$.

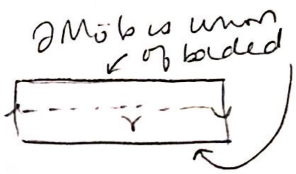
eg. Klein bottle = union of 2 Möbius bands glued along their S^1 ∂s.







See back for alternate polygon construction

Each Möb retracts onto its core circle γ 

(Thicken the ∂s of the two Möb bands a little so that $A \cap B$ is a circle $\times (-\varepsilon, \varepsilon)$, so $\mathring{A} \cup \mathring{B} = $ Klein.)

Reduced M-V:

$$0 \longrightarrow H_2(K) \ (?)$$

$$\hookrightarrow \underset{\mathbb{Z}}{H_1(A \cap B)} \xrightarrow{\Phi} \underset{\mathbb{Z} \oplus \mathbb{Z}}{H_1(A) \oplus H_1(B)} \xrightarrow{\Psi} H_1(K)$$

$$\hookrightarrow 0$$

$\underbrace{\Phi(\partial \text{Möb}) = (2\gamma, -2\gamma)}$ $\quad (\deg(\partial\text{Möb} \to \gamma) = 2)$

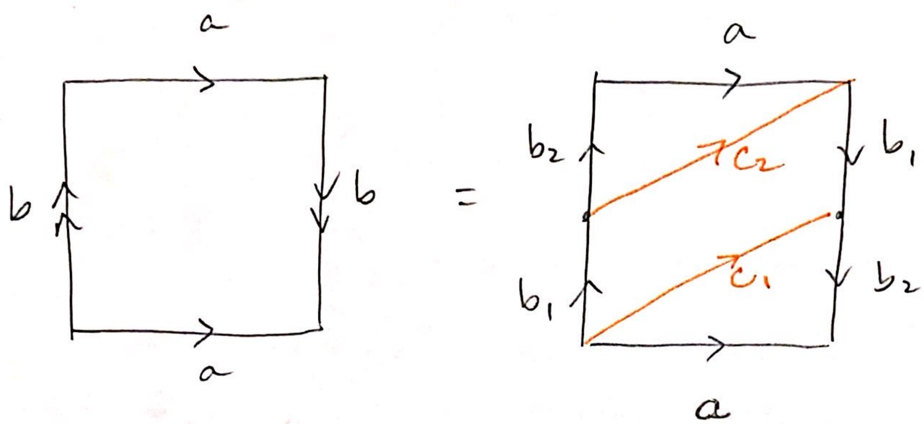
injective $\Rightarrow$ $H_2(K) \cong 0$.

$\Rightarrow \ker \Psi = \langle (2\gamma, -2\gamma) \rangle$

Ψ surjective $\Rightarrow$ $H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

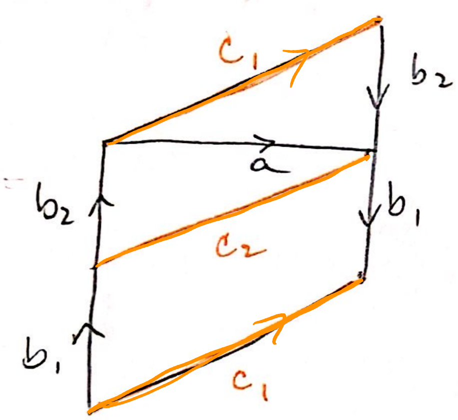
(use bases $(1,0), (1,-1)$ for $\mathbb{Z} \oplus \mathbb{Z} \cong H_1(A) \oplus H_1(B)$)

If we use the other square diagram:



$c = c_1 c_2$

=



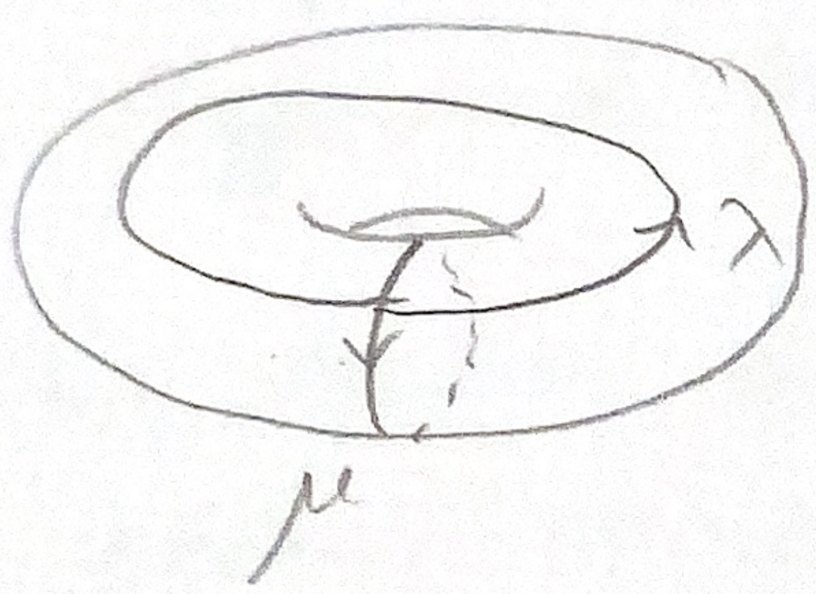
= 2 Mobius bands glued along $c = c_1 c_2$

Lecture 18 Last Remarks on Homology

- Kunneth Formula
- Homology w/ coeffs.
  ↳ UCT

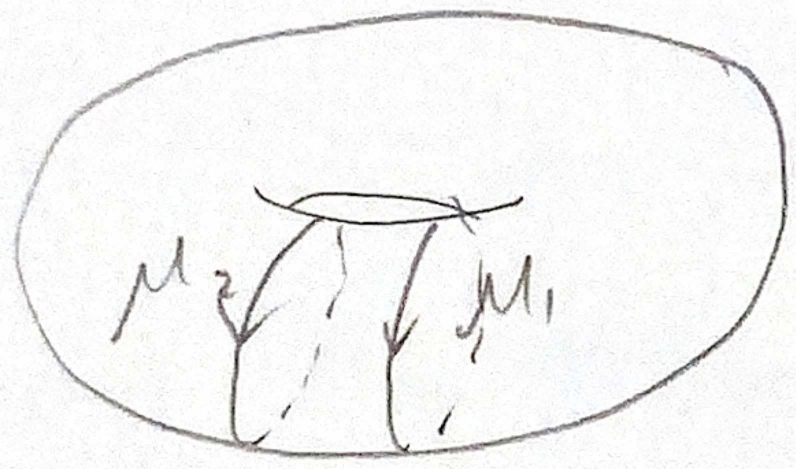
Rmk • homology in low dims as maps of S^1, surfaces into space.

Rmk. Homology classes can be envisioned as maps of some spaces into X (details complicated)

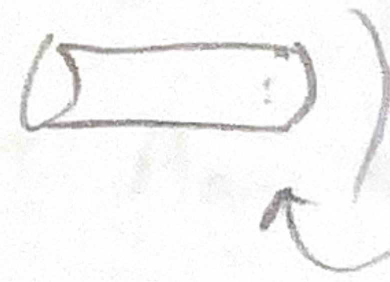


μ = meridian
λ = (a) longitude

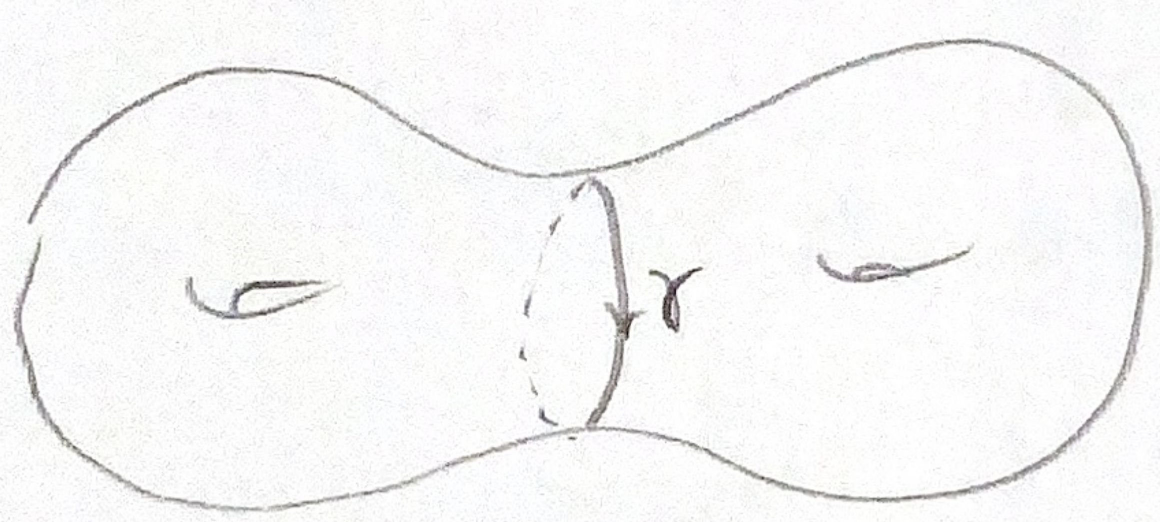
These are lin indep homo classes



$\mu_1 \simeq \mu_2$ in homology because

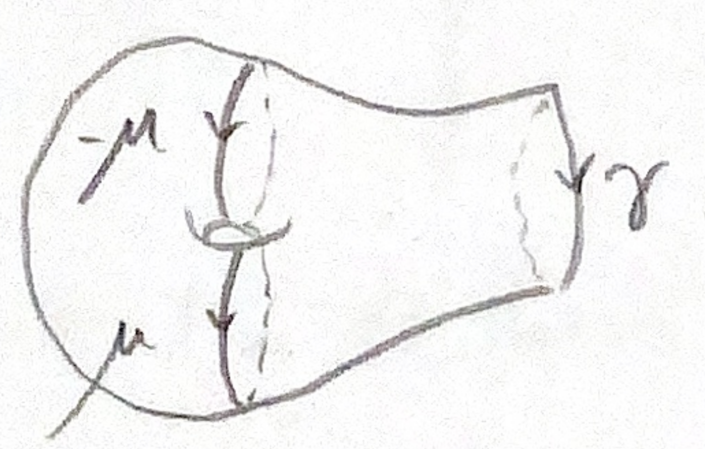
$-\mu_1 \cup \mu_2 = \partial \Big($  $\Big)$ a surface.

Hence



this separating curve is null homologous

In fact, $\simeq$



An algebraic fact

defn. (⊗ of chain cpxs)

$$\mathcal{A} = \cdots \xrightarrow{\partial^{\mathcal{A}}_{n+1}} A_n \xrightarrow{\partial^{\mathcal{A}}_n} A_{n-1} \xrightarrow{\partial^{\mathcal{A}}_{n-1}} \cdots$$

$$\mathcal{B} = \cdots \xrightarrow{\partial^{\mathcal{B}}_{n+1}} A_n \xrightarrow{\partial^{\mathcal{B}}_n} A_{n-1} \xrightarrow{\partial^{\mathcal{B}}_{n-1}} \cdots$$

Let $\mathcal{A}$, $\mathcal{B}$ be two ~~(reasonably bounded)~~ complexes - we will assume these are supported in nonnegative hom degrees.

The tensor product $\mathcal{A} \otimes \mathcal{B}$ is the flattened chain complex of

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & A_{n+1} \otimes B_{n+1} \arrow[r] \arrow[d, "id_A \otimes (-1)^{n+1} \partial^{\mathcal{B}}_{n+1}"] & A_n \otimes B_{n+1} \arrow[r] \arrow[d] & A_{n-1} \otimes B_{n+1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & A_{n+1} \otimes B_n \arrow[r, "\partial^{\mathcal{A}}_n \otimes id_B"] \arrow[d, "id_{\mathcal{A}} \otimes (-1)^n \partial^{\mathcal{B}}_{n}"] & A_n \otimes B_n \arrow[r] \arrow[d] & A_{n-1} \otimes B_n \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & A_{n+1} \otimes B_{n-1} \arrow[r] \arrow[d, "id_A \otimes (-1)^{n-1} \partial^{\mathcal{B}}_{n-1}"] & A_n \otimes B_{n-1} \arrow[r] \arrow[d] & A_{n-1} \otimes B_{n-1} \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} &
\end{tikzcd}$$

$$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j \qquad \partial^{\otimes}(a \otimes b) = \partial^{\mathcal{A}}_i a \otimes b + (-1)^i (a \otimes \partial^{\mathcal{B}}_j b)$$

Rmks

① boundedness needed for finite sums in ∂

↳ can accommodate differently bdded cpxs w/ bddness on one complex, etc or expanding the category you are working with

② can check $(\partial^{\otimes})^2 = 0$ by direct computation (& ∂^A, ∂^B are diffe's)

$\boxed{\text{Thm 1}}$ Suppose A and B are free. Then there is a canonically defined SES $\forall n$:

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(A) \otimes H_j(B) \longrightarrow H_n(A \otimes B)$$

$$\longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}\big(H_i(A), H_j(B)\big) \longrightarrow 0$$

which yields (noncanonical) isomorphisms

$$H_n(A \otimes B) \cong \bigoplus_{i+j=n} H_i(A) \otimes H_j(B) \oplus \bigoplus_{i+j=n-1} \mathrm{Tor}\big(H_i(A), H_j(B)\big).$$

Rmk ① There is a proof in Homotopical Topology by Fomenko - Fuchs

② What is Tor? (In our case Tor just means Tor_1 ; $\mathrm{Tor}_i^{\mathbb{Z}} = 0$ for $i \geq 2$)
$\mathrm{Tor}(\underbrace{A, B}_{\mathbb{Z}\text{-mods}})$

We work with $\mathbb{Z}$-modules, recall.

- If A or B is free, then $\mathrm{Tor}(A, B) = 0$
- (In fact converse also true)

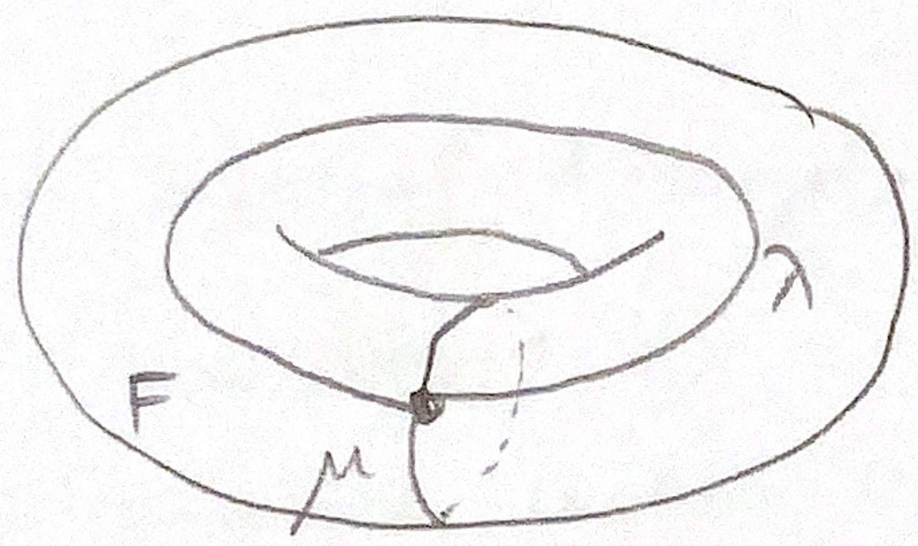
Künneth's Formula (thm)

Apply thm 1. to CW-homology of two cell complexes X and Y:

- indeed $C_\cdot(X)$ and $C_\cdot(Y)$ are free.
- we can identify the pure tensors

$$e^i_X \otimes e^j_Y \quad \text{with the cell } e^i_X \times e^j_Y$$
in the Cartesian product

eg.



$$S^1 \times S^1 = T^2$$
$$\mu \times \lambda = F$$
$$\mu \times \bullet = \text{“}\mu\text{”}$$
$$\bullet \times \lambda = \text{“}\lambda\text{”}$$

Therefore we get a noncanonical isom:

$$\boxed{\underbrace{H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \oplus \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y))}_{\text{“Künneth Formula”}}}$$

Homology with coefficients

(later... UCT (3.A))

Can define homology over other coeffs than $\mathbb{Z}$ (abel gps)

Let G be an abelian group. (a $\mathbb{Z}$-module)

- let $C_n(X;G) = C_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} G$ $\left(\{e_\alpha\}_{\alpha \in A} = \text{cells} \quad \cong \bigoplus_{\alpha \in A} G \right)$
- When building cellular cpx, augment w/ G instead of $\mathbb{Z}$ (actually $\mathbb{Z} \otimes_{\mathbb{Z}} G$).

The resulting $H_n(X;G)$ are the homology groups of X with coeffs in X.

Really useful example $G = \mathbb{Z}/2\mathbb{Z}$

- no signs ☺
- works with non orientable spaces more intuitively

e.g. 1 $\mathbb{RP}^n$ over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. (see iPad notes for calculation) → not shared yet HW?

$$H_k(\mathbb{RP}^n; \mathbb{F}_2) = \begin{cases} \mathbb{F} & k = 0, \dots, n \\ 0 & \text{o/w} \end{cases}$$

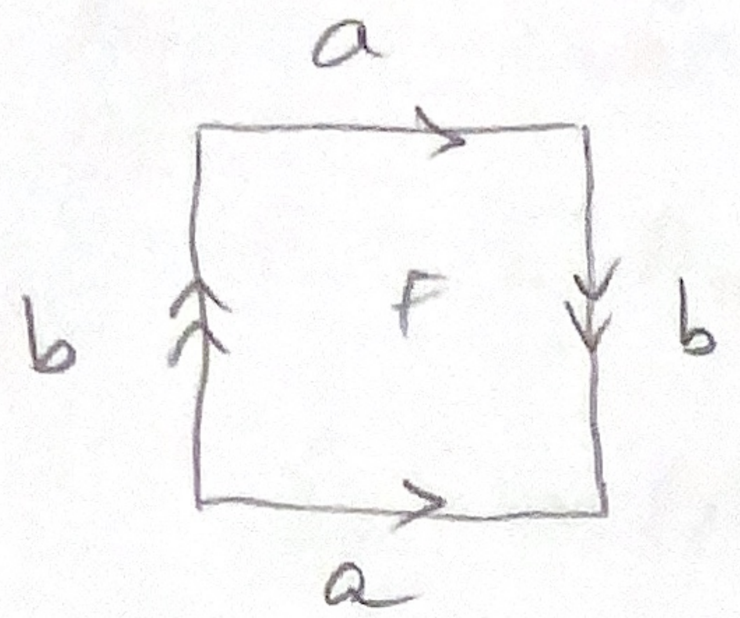
where as

$$H_k(\mathbb{RP}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2 & k = 1 \\ 0 & k = 2 \quad \leftarrow \text{particularly disturbing} \\ 0 & \text{o/w.} \end{cases}$$

Rmk. Nevertheless, there is a Universal Coeff thm that says all hom. gps are det by those w/ $\mathbb{Z}$-coeffs.

eg 2. Klein bottle (we used Mayer Vietoris)

{* remark from last time}



Over $\mathbb{F}_2$:

$d_0 = 0$

$d_1 = 0$

$d_2(F) = 2a = 0$

recall I write $\mathbb{F}_2$ but is actually the $\mathbb{Z}$-mod $\mathbb{Z}/2\mathbb{Z}$

$\Longrightarrow$ Homology is same as chain cpx

$$0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{F}_2 \oplus \mathbb{F}_2 \longrightarrow \mathbb{F}_2 \longrightarrow 0 \qquad (!!)$$

- Compare with T^2 (now on equal footing)
- Also compare w/ what we got from Mayer-Vietoris:

$$H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z} \oplus \mathbb{Z}/2, \quad H_2 = 0 \qquad (\ast)$$

(HW?)

Q: Can we recover (!!) from $(\ast)$? – note we just did some tensoring...

If we unwrap G into its free resolution, we satisfy the conditions of thm 1!

Some algebra leads to the following very useful theorem:

> **thm** (Universal Coeff thm for homology)
>
> $$0 \to H_i(X;\mathbb{Z}) \otimes G \longrightarrow H_i(X;G) \to \mathrm{Tor}(H_{i-1}(X;\mathbb{Z}), G) \to 0$$