Week 6 lectures (19-21) (PART I)

Cohomology is algebraically very closely related to homology.

homology: chain complex:

$$\mathcal{C} = \left(\cdots \longrightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} \underline{C_0} \longrightarrow 0. \right)$$

cohomology: cochain cpx: Over $G \in \mathbb{Z}$-mod coefficients:

$$0 \to \underline{\mathrm{Hom}_{\mathbb{Z}}(C_0, G)} \xrightarrow{\delta} \mathrm{Hom}_{\mathbb{Z}}(C_1, G) \xrightarrow{\delta} \mathrm{Hom}_{\mathbb{Z}}(C_2, G) \longrightarrow \cdots$$

$$0 \longrightarrow \underline{C_0}^* \xrightarrow{\delta} C_1^* \xrightarrow{\delta} C_2^* \longrightarrow \cdots$$ (everything dualized)

As a result of dualizing, the cohomology functor

$$\left\{ \begin{matrix} \text{Top spaces,} \\ \text{cts. maps} \end{matrix} \right\} \xrightarrow{H^*(-)} \text{graded } \mathbb{Z}\text{-mod}$$

is <u>contravariant</u>:

a map of spaces $f : X \to Y$ determines

a map $f^* : H^*(Y) \longrightarrow H^*(X)$ on cohomology.

Contravariance allows us to define the cup product, giving cohomology an algebra structure!

The cohomology groups $H^*(\mathscr{C})$ are determined by the homology groups (over $\mathbb{Z}$) via a universal coefficient theorem:

<u>thm 3.2</u> (Universal Coefficient Theorem for Cohomology)

If a chain cpx $\mathscr{C}$ of free abelian groups has homology groups $H_n(\mathscr{C})$ (ie over $\mathbb{Z}$), then the cohomology groups $H^n(\mathscr{C}; G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ are determined by split exact sequences

$$0 \to \mathrm{Ext}(H_{n-1}(\mathscr{C}), G) \to H^n(\mathscr{C}; G) \to \mathrm{Hom}(H_n(\mathscr{C}), G) \to 0$$

Treat Ext as black box just as we did with Tor.

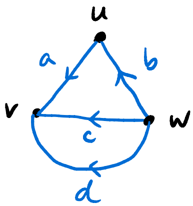
<u>Useful Properties of $\mathrm{Ext}(H, G)$:</u>

- $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$
- $\mathrm{Ext}(H, G) = 0$ if H is free
- $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$.

What does it mean to dualize? (Idea of cohomology)

↳ we haven't discussed how to get δ from ∂ yet!

(Start: idea from simplicial cohomology, ≡ singular cohom. §3.1.)

Let X =  a 1D Δ-complex.

Let G be an abelian group (our coeffs group).

- $\Delta^0(X;G) = \mathrm{Hom}(\Delta_0(X), G)$
  $= \text{fns } \{u, v, w\} \longrightarrow G.$
- $\Delta^1(X;G) = \mathrm{Hom}(\Delta_1(X), G)$
  $= \text{fns } \{a, b, c, d\} \longrightarrow G.$

abelian group

- $\delta : \Delta^0(X;G) \longrightarrow \Delta^1(X;G)$

$$\varphi \longmapsto \delta\varphi := \varphi \circ \partial$$

$$\varphi \circ \partial([v_0, v_1]) = \varphi(v_1 - v_0)$$
$$= \varphi(v_1) - \varphi(v_0)$$

Analogy: φ assigns to each point an elevation / energy. $\delta\varphi$ measures the net change in elevation over an edge.

• $H^0(X;G) \cong \ker\delta \subset \Delta^0(X;G)$

$\varphi \in \ker\delta$ iff $\varphi\circ\partial(e) = 0$ $\forall$ edges e of X.

iff φ is constant on each cpnt of X.

$\Longrightarrow H^0(X;G) = \prod_{\substack{\text{path}\\\text{cpnts}\\\text{of } X}} G$

$\mathrm{Hom}(\bigoplus M_i, G)$
$\cong \prod \mathrm{Hom}(M_i, G)$

• $H^1(X;G) \cong \Delta^1(X;G)/\operatorname{im}\delta$

⚠ our example is a 1-complex.

$\psi \in \Delta^1(X;G)$ is in $\operatorname{im}\delta$

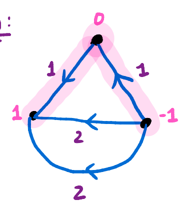
iff $\exists\, \varphi \in \Delta^0(X;G)$ s.t. $\delta\varphi = \varphi\circ\partial = \psi$.

Does the set of elevation changes here a solution (values you can write on the vertices?)

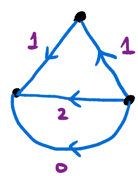
($G = \mathbb{Z}$:)

Yes:

φ



No:



<u>Observe</u> $\delta\varphi = \psi$ is always solvable if X is a tree.

So for the edges outside a maximal tree, we have degrees of freedom, giving us an idea of $H_1(X;G)$:

$$H_1(X;G) \cong \prod_{\substack{e \text{ not} \\ \text{in maximal} \\ \text{tree}}} G$$

<u>Aside</u> $\mathrm{Hom}(\bigoplus M_i, \mathbb{Z}) \cong \prod \mathrm{Hom}(M_i, \mathbb{Z})$.

You can read about the 2D case in Hatcher.

Similar idea, but now use topographical maps

Again, "elevation" is used as analogy for assigning an element of G to each vertex.

Cohomology of spaces

X = space $\quad G$ = abelian group

$C^n(X;G) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X), G)$ singular n-cochains with coefficients in G

$\varphi \in C^n(X;G)$ assigns to each $\sigma: \Delta^n \to X$ a value $\varphi(\sigma) \in G$.

$$\delta = \partial^*: C^n(X;G) \to C^{n+1}(X;G) \qquad \text{coboundary map}$$
$$\varphi \longmapsto \varphi \circ \partial$$

Explicitly,

$$\delta\varphi(\sigma) = \sum_i (-1)^i \varphi\left(\sigma\big|_{[v_0, \dots, \hat{v}_i, \dots, v_{n+1}]}\right)$$

learn about a space by considering functions on the space...

- $\delta^2 = 0$ because $\partial^2 = 0$
- $\ker\delta$ = cocycles, $\mathrm{im}\,\delta$ = coboundaries.

  $\varphi \in \ker\delta$ if $\delta\varphi = \varphi\partial = 0$, ie $\varphi = 0$ on boundaries

Rmk. $0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}), G) \to H^n(\mathcal{C}; G) \longrightarrow \mathrm{Hom}(H_n(\mathcal{C}), G) \to 0$

- When $n=0$, we have $H^0(X;G) \cong \mathrm{Hom}(H_0(X);G)$

  As we saw in the low-dim'l example, $\varphi \in \ker\delta$ iff it is a constant fn. on path cpnts.

  So $H^0(X;G) =$ fns from path cpnts of X to G

  $$\cong \prod_{\substack{\text{path}\\ \text{cpnts}\\ \text{of } X}} G$$

- When $n=1$, $\mathrm{Ext}(H_0(X), G) = 0$ since $\overbrace{H_0(X)}^{\text{homology}}$ is free.

  So we also have $H^1(X;G) \cong \mathrm{Hom}(H_1(X), G)$.

  $\cong \mathrm{Hom}(\pi_1(X), G)$, since G is already abelian

With field coefficients   Let F = field.

Recall $C_n(X;F) = C_n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} F$

- As F-module, has basis the singular n-simplicies of X.

- $\mathrm{Hom}_F(C_n(X;F), F) \overset{\text{as abelian gps}}{\cong} \mathrm{Hom}_{\mathbb{Z}}(C_n(X), F)$

  since for both you just specify the values on the basis: singular simplicies

- By a generalization of the UCT to modules over a PID (rather than $\mathbb{Z}$),

  we get

$$H^n(X;F) \cong \mathrm{Hom}_F(H_n(X;F), F)$$

  $\mathrm{Ext}_F(-,F)$ are all 0 since F is a field and all modules are free

Therefore: over a field, cohomology is exactly dual to homology.

homology: chain complex:

$$\mathscr{C} = \left(\cdots \longrightarrow C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} \underline{C_0} \longrightarrow 0. \right)$$

cohomology: cochain cpx: Over $G \in \mathbb{Z}$-mod coefficients

$$0 \longrightarrow \underline{\mathrm{Hom}_{\mathbb{Z}}(C_0, G)} \xrightarrow{\delta} \mathrm{Hom}_{\mathbb{Z}}(C_1, G) \xrightarrow{\delta} \mathrm{Hom}_{\mathbb{Z}}(C_2, G) \longrightarrow \cdots$$

$$0 \longrightarrow \underline{C_0}^* \xrightarrow{\delta} C_1^* \xrightarrow{\delta} C_2^* \longrightarrow \cdots$$

(everything dualized)

We now think through all the constructions from homology and realize they all make sense or have analogous statements in cohomology.

Reduced Cohomology groups

$$\cdots \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

dualize

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{\varepsilon^*} \mathrm{Hom}(C_0(X), G) \xrightarrow{\delta} \cdots$$

$$(f : \mathbb{Z} \to G) \longmapsto (f \circ \varepsilon : C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{f} G)$$

<u>note</u> $f \circ \varepsilon(\sigma) = f(1)$

$$0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}), G) \to H^n(\mathcal{C}; G) \longrightarrow \mathrm{Hom}(H_n(\mathcal{C}), G) \to 0$$

$$0 \to \mathrm{Ext}(\underbrace{\tilde{H}_{-1}(X)}_{=0}, G) \to \tilde{H}^0(X; G) \longrightarrow \mathrm{Hom}(\tilde{H}_0(X), G) \longrightarrow 0$$

$$\Rightarrow \quad \tilde{H}^0(X; G) \cong \mathrm{Hom}(\tilde{H}_0(X), G)$$

Interpretation of $\tilde{H}^0(X;G)$:

from last time:

$$\tilde{H}^0(X;G) = \frac{\ker \delta_0}{\operatorname{im} \varepsilon^*}$$

$\{\varphi : C_0(X) \to G \mid \text{constant on path cpnts}\}$

$\varphi \in \operatorname{im} \varepsilon^*$ if $\varphi = f \circ \varepsilon$ for some f, ie.

$\varphi(\sigma) = f(1) \; \forall \, \sigma.$ (see note above)

$\Rightarrow \operatorname{im} \varepsilon^* = \{\varphi : C_0(X) \to G \mid \varphi \text{ is const on all } \sigma\}$

$$\Rightarrow \tilde{H}^0(X;G) = \frac{\{\varphi : C_0(X) \to G \mid \text{constant on path cpnts}\}}{\{\varphi \text{ that are constant on all of } X\}}$$

Relative groups, LES of a pair.

- Start with SES for pair (X, A)

This SES in a way defines $C_n(X,A)$

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \to 0$$

Now apply $\mathrm{Hom}(-,G)$.

- Note A priori, $(-)^* = \mathrm{Hom}(-,G)$ is left-exact i.e.

  If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact,

  then $0 \to C^* \xrightarrow{\beta^*} B^* \xrightarrow{\alpha^*} A^*$ is exact

  (but $B^* \xrightarrow{\alpha^*} A^*$ might not be surjective)

- But in our case,

  $\mathrm{Hom}(C_n(X,A), G)$

$$0 \to C^n(X,A;G) \xrightarrow{j^*} C^n(X;G) \xrightarrow{i^*} C^n(A;G) \to 0$$

subgroup! (restriction)

Is indeed exact:

Let $\varphi \in C^n(X;G)$. Then $i^*(\varphi) = \varphi|_A$.

Every $\psi : C_n(A) \to G$ can be extended by 0 to a function $\overline{\psi} : C_n(X) \to G$. Then $i^*(\overline{\psi}) = \psi$.

So i^* is indeed surjective.

- We can view $C^n(X,A;G)$ as functions

$$\{\text{singular } n \text{ simplicies in } X\} \longrightarrow G$$

that vanish on simplices in A.

- Relative coboundary maps $\delta : C^n(X,A;G) \to C^{n+1}(X,A;G)$ are restrictions of the absolute $\delta : C^n(X;G) \to C^{n+1}(X;G)$ (to fns that vanish on simplicies in A).

So the same construction from SES $\rightsquigarrow$ LES goes through:

$$\begin{tikzcd}
\cdots H^n(X,A;G) \arrow[r, "j^*"] & H^n(X;G) \arrow[r, "i^*"] & H^n(A;G) \arrow[dll, "\delta"'] \\
H^{n+1}(X,A;G) \arrow[r] & \cdots
\end{tikzcd}$$

To compute the connecting map δ:

$$\delta[\varphi] = [\bar{\varphi}\partial].$$

extended

$$\begin{tikzcd}
 & & C^{n+1}(X,A;G) \arrow[d, "j^*"] & \bar{\varphi}\partial \arrow[d, mapsto] \\
\bar{\varphi} \arrow[rrr, mapsto, bend left=10] \arrow[d, mapsto] & C^n(X;G) \arrow[r] \arrow[d, "i^*"'] & C^{n+1}(X;G) & \bar{\varphi}\partial \\
\varphi & C^n(A;G) \arrow[uur, dashed] & &
\end{tikzcd}$$

$\varphi\partial = 0$

- There is a relationship b/w the ~~cohomology~~ connecting maps!

$$\begin{tikzcd}
H^n(A;G) \arrow[r, "\delta"] \arrow[d, "h"'] & H^{n+1}(X,A;G) \arrow[d, "h"] \\
\mathrm{Hom}(H_n(A),G) \arrow[r, "\partial^*"] & \mathrm{Hom}(H_{n+1}(X,A),G)
\end{tikzcd}$$

$$0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}),G) \to H^n(\mathcal{C};G) \xrightarrow{h} \mathrm{Hom}(H_n(\mathcal{C}),G) \to 0$$

Induced Homomorphisms

H^* is a contravariant functor.

$f: X \to Y$

$\hookrightarrow$ $f_\#: C_n(X) \to C_n(Y)$ $\quad$ $\mathrm{Hom}(-,G)$

$\hookrightarrow$ $f^\#: C^n(Y;G) \to C^n(X;G)$

induces $f^*: H^n(Y;G) \to H^n(X;G)$

- $\mathbb{1}^\# = \mathbb{1}$ so $\mathbb{1}^* = \mathbb{1}$.
- $(fg)^\# = g^\# f^\#$ so $(fg)^* = g^* f^*$

Homotopy Invariance

Recall we found a homotopy (prism operator) P s.t.

$$g_\# - f_\# = \partial P + P\partial.$$

Dualizing this relation we get

$$g^\# - f^\# = P^*\partial^* + \partial^* P^* = P^*\delta + \delta P^*$$

so P^* a chain htpy b/w $f^\#$ and $g^\#$.

$\Rightarrow$ they are chain htpic $\Rightarrow$ $f^* = g^*$ on cohom.

Excision

thm. Suppose $Z \subset A \subset X$ with $\bar{Z} \subset \mathring{A}$.

Then $i : (X-Z, A-Z) \hookrightarrow (X, A)$ induces isomorphisms

$$i^* : H^n(X, A; G) \longrightarrow H^n(X-Z, A-Z; G) \quad \forall n.$$

Recall in the pf we had $\rho i = \mathbb{1}$ and $\mathbb{1} - i\rho = \partial D + D\partial$.
Again dualize + use the 5-lemma.

Mayer-Vietoris Sequences

We still have the SES of duals

$$0 \to C^n(A+B; G) \longrightarrow C^n(A;G) \oplus C^n(B;G) \longrightarrow C^n(A \cap B; G) \longrightarrow 0$$

$$f \longmapsto (f|_A, f|_B)$$

$$(f, g) \longmapsto f|_{A\cap B} - g|_{A \cap B}$$

Recall: $C_n(A+B) \subset C_n(X)$, where inclusion is a chain htpy equivalence.

$$C^n(A+B; G) = \mathrm{Hom}(C_n(A+B), G)$$

Taking duals, the restriction map

$C^n(X; G) \xrightarrow{i^*} C^n(A+B; G)$ is also a chain htpy equiv.

Lecture 21

Cross product in homology

$$H_i(X) \times H_j(Y) \longrightarrow H_{i+j}(X \times Y)$$

Induced by, in the CW cpx case,

$$C_i(X) \times C_j(Y) \longrightarrow C_{i+j}(X \times Y)$$

$$(e^i_\alpha, e^j_\beta) \longmapsto e^i_\alpha \times e^j_\beta$$

(Give $X \times Y$ the product cell structure)

But if we want to turn $H_*(X)$ into a ring, we need some natural map

$$H_{i+j}(X \times X) \longrightarrow H_{i+j}(X).$$

There isn't really a natural choice of map $X \times X \to X$ here.

On the other hand there is a very natural diagonal map $\Delta : X \to X \times X$

So we want a contravariant construction instead.

we can also define a cross product for CW cochains:

$$C^i(X) \times C^j(Y) \longrightarrow C^{i+j}(X \times Y)$$
$$(\varphi, \psi) \longmapsto \text{“}\varphi \times \psi\text{”} \text{ where}$$

$$(\varphi \times \psi)(e^i_\alpha \times e^j_\beta) = \varphi(e^i_\alpha) \cdot \psi(e^j_\beta)$$

multiplication in the ring $\mathbb{Z}$

We could check that this induces a product on cohomology.

$$H^i(X) \times H^j(Y) \xrightarrow{\times} H^{i+j}(X \times X)$$

The composition

$$\begin{tikzcd} H^i(X) \times H^j(Y) \arrow[r, "\times"] \arrow[rr, bend left, "\smile"] & H^{i+j}(X \times X) \arrow[r, "\Delta^*"] & H^{i+j}(X) \end{tikzcd}$$

\smile

turns out to be the <u>cup product</u>

Note that we have not checked many details. Most importantly, invariance under choice of cell structure!

Direct construction of cup product

Let R be a ring (eg. $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Q}$)

Cup product on cochains:

Let $\varphi \in C^k(X;R)$

$\psi \in C^\ell(X;R)$

$\sigma : \Delta^{k+\ell} \to X$ generator of $C_{k+\ell}(X)$

Define

$$(\varphi \smile \psi)(\sigma) = \varphi\left(\sigma|_{[v_0,\dots,v_k]}\right) \cdot \psi\left(\sigma|_{[v_k,\dots,v_{k+\ell}]}\right)$$

mult in R
(assoc, distributive)

Relate with differentials:

Lemma 3.6 $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$.

Pf. Let $\sigma : \Delta^{k+\ell+1} \longrightarrow X$.

- $(\delta\varphi \smile \psi)(\sigma)$

$$= \sum_{i=0}^{k+1} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,\hat{v_i},\dots,v_{k+1}]}\right) \cdot \psi\left(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}\right)$$

last term:

$$(-1)^{k+1} \varphi\left(\sigma|_{[v_0,\dots,v_k]}\right) \cdot \psi\left(\sigma|_{[v_{k+1},\dots,v_{k+\ell+1}]}\right)$$

- $(-1)^k(\varphi \smile \delta\psi)(\sigma)$

$$= \sum_{i=k}^{k+\ell+1} (-1)^i \varphi\left(\sigma|_{[v_0,\dots,v_k]}\right) \cdot \psi\left(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+\ell+1}]}\right)$$

w/ last term $(-1)^k \cdot \star$

These two terms cancel in the sum, and the remainder is $\delta(\varphi \smile \psi)(\sigma) = (\varphi \smile \psi)(\partial\sigma)$. $\square$

lemma 3.6 $\Rightarrow$

① product of two cocycles is a cocycle:

If $\delta\varphi, \delta\psi = 0$ then $\delta(\varphi \smile \psi) = \cancelto{0}{\delta\varphi} \smile \psi \pm \varphi \smile \cancelto{0}{\delta\psi} = 0$

② product of cocycle and coboundary $= 0$ (either order)

- If $\delta\varphi = 0$ then $\varphi \smile \delta\psi = \pm\,\delta(\varphi \smile \psi)$

- If $\delta\psi = 0$ then $\delta\varphi \smile \psi = \delta(\varphi \smile \psi)$.

By ① & ②, $\smile$ descends to a cup product on cohomology:

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R)$$

$$([\varphi],[\psi]) \longmapsto [\varphi \smile \psi]$$

$(\varphi + \delta\varphi') \smile (\psi + \delta\psi') = \varphi \smile \psi +$ coboundary

This gives cohomology a graded ring structure, where the identity element is $1 \in H^0(X;R)$ rep'd by the cochain that sends all 0-simplicies to the identity $1 \in R$.