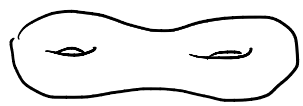
Lecture 22 5.19.25

Recall $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$, $\sigma : \Delta^{k+\ell} \to X$

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\,\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})$$

Example $\Sigma_2 =$  (can generalize to Σ_g)

① We know $H_*(\Sigma_2)$ is $H_0, H_1, H_2 \cong \mathbb{Z}, \mathbb{Z}^4, \mathbb{Z}$, all free.

UCT $\Rightarrow$ $H^i(\Sigma_2) \cong \mathrm{Hom}(H_i(\Sigma_g), \mathbb{Z})$.

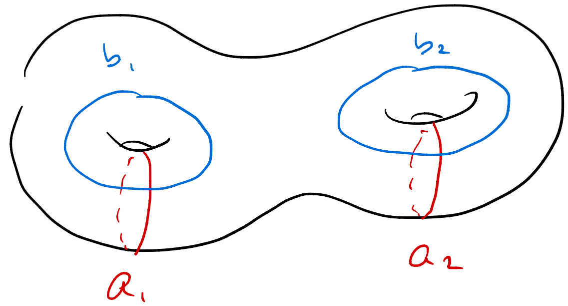
② Using Hatcher's notation, we'll write α_i as the dual to a_i, β_i dual to b_i; etc.

I personally prefer a_i^*; see HW. You can of course use whatever you want.

③ $H_0(\Sigma_2) \times H_0(\Sigma_2) \xrightarrow{\smile} H_0(\Sigma_2)$ is not very interesting; it's just multiplication in $\mathbb{Z}$.

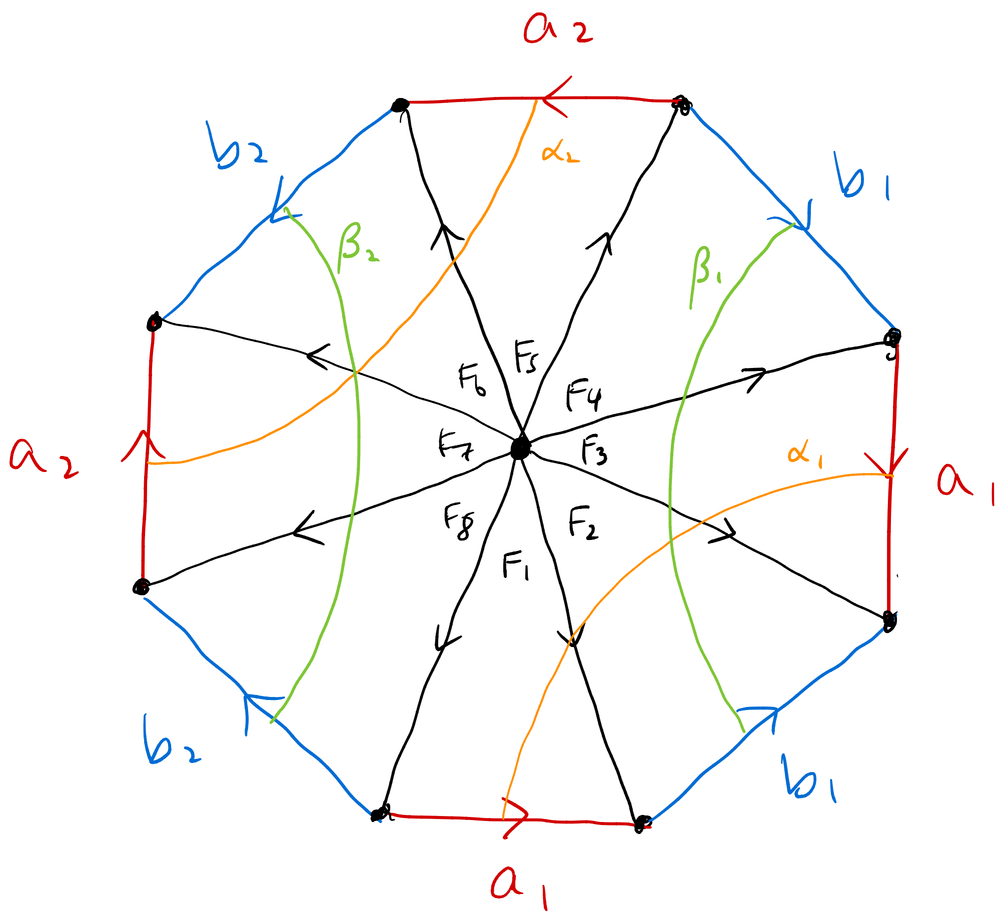
④ $H_1(\Sigma_2) \times H_1(\Sigma_2) \xrightarrow{\smile} H_2(\Sigma_2)$ is the only interesting cup product as $H_k = 0$ for $k > 2$.

Here we know $H_1(\Sigma_2)$ is generated by a_1, a_2, b_1, b_2



or better yet let's rep by the homologous cycles from the polygon description:

Simplicial structure on Σ_2 (that generalizes to Σ_g)



(The black edges do not represent classes in homology, as we already know from CW homology.)

⑤ We now depict the duals $\alpha_i, \beta_i \in H^1(\Sigma_2; \mathbb{Z})$ by curves that transversely intersect a_i, b_i (respectively) exactly once.

α_i: β_i drawn. — these are meant to rep. cohomology classes but by abuse of notation also denote the curves.

we can define simplicial cocycles φ_i, ψ_i representing the cohomology classes α_i, β_i resp.'ly by declaring

- The value of φ_i on a (simplicial) chain y $=$ # times y intersects α_i
  - actually should give α_i a transverse orientation and count signed intersection, but note that the way it's drawn, the intersecting curves all travel the same way across the curve α_i.

- Similarly for ψ_i and β_i.

AND checking that these are actually cocycles!

The cocycle condition: $\delta\varphi_i = 0$ $\quad \delta\psi_i = 0$.

$$\delta\varphi\left([v_0, v_1, v_2]\right)$$
$$= \varphi\left([v_0, v_1]\right) - \varphi\left([v_1, v_2]\right) + \varphi\left([v_0, v_2]\right) = 0$$

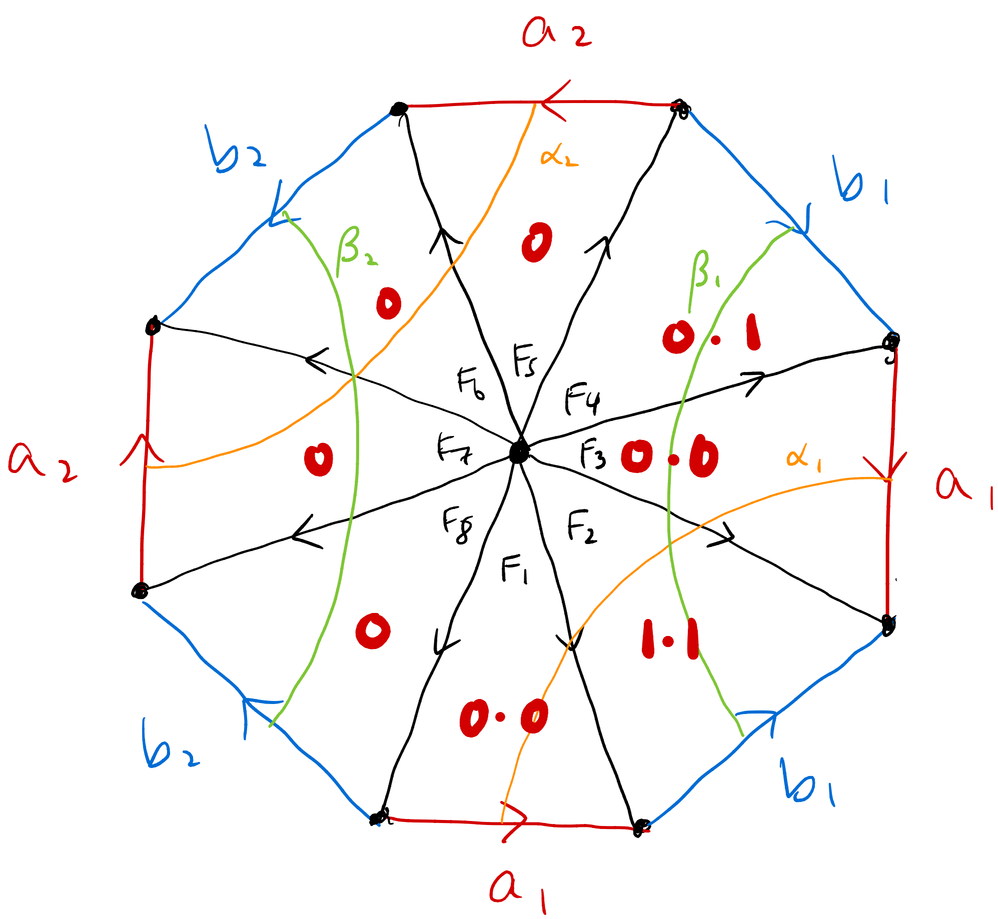
So the cocycle condition becomes

$$\varphi\left([v_0, v_1]\right) + \varphi\left([v_0, v_2]\right) = \varphi\left([v_1, v_2]\right). \quad \text{(*)}$$

* check now that φ_i, ψ_i are all cocycles indeed. (we are lucky).
The α_i, β_i always $\cap$ a $[v_0, v_2]$, + then one else on ∂ of a Δ^2.

⑥ Now compute cup product at the cochain level.

* compute $\varphi_1 \smile \varphi_1$ on each simplex. straight from definition of $\smile$.



So $\varphi_1 \smile \varphi_1$ evaluates to 1 only on simplex F_2

What do we do w/ this info?

(7) We need to find a representative of a generator of H_2.

Easy - we know from cellular homology that

$c = F_1 + F_2 - F_3 - F_4 + F_5 + F_6 - F_7 - F_8$

is a generator.

Then $(\varphi_1 \smile \psi_1)(c) = 1$. $(\varphi_1 \smile \psi_1 = c^*)$

so if γ is the dual to $[c]$

recall $H^2(\Sigma_2) \cong \mathrm{Hom}(H_2(\Sigma_2), \mathbb{Z})$

in our present example

then $\alpha_1 \smile \beta_1 = \gamma$.

Since $[c]$ is a generator of $H_2(\Sigma_2)$,

γ is a generator of $H^2(\Sigma_2)$

(8) Let's compute the rest:

- $\psi_1 \smile \varphi_1 = ?$ only $= 1$ on F_3.

$\Rightarrow (\psi_1 \smile \varphi_1)(c) = -1$

$\Rightarrow \beta_1 \smile \alpha_1 = -\gamma = -\alpha_1 \smile \beta_1$

rmk. This turns out to generalise - if R is commutative, then

if $\alpha \in H^k(X;R)$, $\beta \in H^l(X;R)$ then

$$\beta \smile \alpha = (-1)^{k \cdot l} \alpha \smile \beta.$$

+ depends only on dimension (ie "degree")

overloaded term here.

// Lect 22.

Lect 23: draw octagon, start @ point (9).

- $\alpha_2 \smile \beta_2 = \gamma$, $\beta_2 \smile \alpha_2 = -\gamma$ by symmetry
- $\alpha_1 \smile \alpha_1 = 0$ (compute together on picture)
- same w/ all the α_i, β_i; $\alpha_i \smile \alpha_j$, etc.

(9) Conclude:

$$\alpha_i \smile \beta_j = \begin{Bmatrix} \gamma & i=j \\ 0 & o/w \end{Bmatrix} = -\beta_i \smile \alpha_j$$

$$\alpha_i \smile \alpha_j = 0, \quad \beta_i \smile \beta_j = 0.$$

Rmk. Observe that the cup product of two dual curves is only nonzero when they intersect!

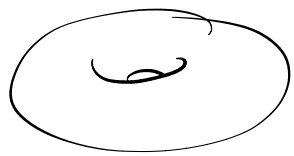
↪ think about why this is true on a Δ^2
- they both need to leave the Δ^2 along $[v_0, v_2]$.

We can also say this for the same dual curve (see next example) by considering α with its pushoff α^+!

disjoint except at necessary transverse intersections

eg. In this example, α_i^+ is entirely disjoint from α_i
— same w/ the β_i.

(8) significance: the cup product on cohomology distinguishes



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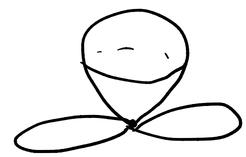


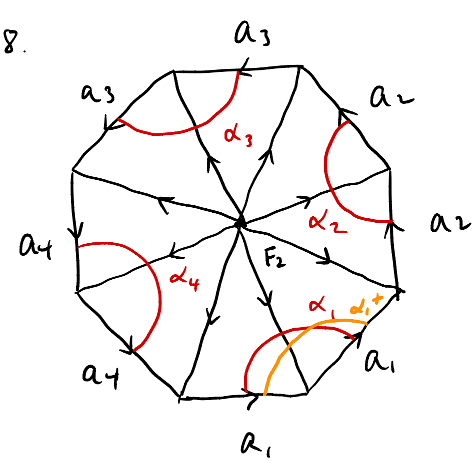
$S^1 \times S^1$ $\qquad$ $S^1 \vee S^2 \vee S^1$

nontrivial cup product. $\qquad$ trivial cup product.

Rmk. Amazing coincidence for prev. example: could find the dual arcs that geometrically encode cocycle representatives

Sometimes we can't, at least over $\mathbb{Z}$ coefficients:

eg. 3.8.



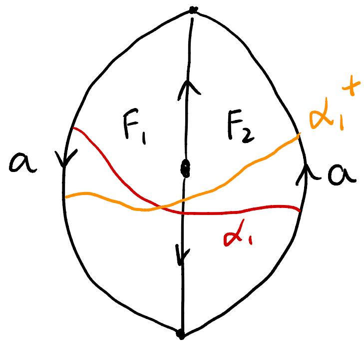
non orientable closed surface of $\chi = 1 - 4 + 1 = -2$ (nonorientable genus 4)

Note that if we define φ_1 on cycles to be "# intersections with α_1" then φ_1 is not a cocycle, because $\delta\varphi_1(F_2) = 1 - 0 + 1 = 2$.

However, if we work over $\mathbb{F}_2$, then $2=0$ so φ_1 is a cocycle!

Let's compute the $\smile$ product for $\mathbb{RP}^2$ over $\mathbb{F}_2$.

eg.



Again define φ_1 on a cycle by the # intersections with α_1, mod 2.

- check that φ_1 is indeed a cocycle

$(\varphi_1 \smile \varphi_1)(F_1) = 0 \cdot 1$

$(\varphi_1 \smile \varphi_1)(F_2) = 1 \cdot 1.$

we are over field coeffs, so cohomology is dual to homology over $\mathbb{F}$ coeffs (recall).

The generator $c = F_1 + F_2$ (intuition from cellular homology)
(ie $[c]$ generates $H_2(\mathbb{RP}^2; \mathbb{F}_2)$)
has dual γ which generates $H^2(\mathbb{RP}^2; \mathbb{F}_2)$.

Since $(\varphi_1 \smile \varphi_1)(c) = 1$, $\boxed{\alpha_1 \smile \alpha_1 = \gamma}$.

In general:

- Σ_g = genus g orientable closed surface

$$\alpha_i \smile \beta_j = \begin{Bmatrix} \gamma & i=j \\ 0 & o/w \end{Bmatrix} = -\beta_i \smile \alpha_j$$

$$\alpha_i \smile \alpha_j = 0, \quad \beta_i \smile \beta_j = 0.$$

- N_k = nonorientable closed surface of $\chi = 2-k$

$$\alpha_i \smile \alpha_i = \gamma, \quad \alpha_i \smile \alpha_j = 0 \text{ when } i \neq j$$

The cohomology ring

Since $\smile$ is an associative, distributive multiplication, we can associate a ring structure to $H^*(X)$!

It will be a graded ring: $H^*(X) = \bigoplus H^i(X)$ where the dimension is the grading.

multiplication is graded too as
$$gr(\alpha \smile \beta) = gr(\alpha) + gr(\beta)$$
, indeed.

The constant function valued at $1 \in R$ represents the identity cohomology class.

Graded-commutative ring (when R commutative)!
$$\beta \smile \alpha = (-1)^{|\alpha||\beta|} \alpha \smile \beta$$

eg. we computed $H^*(\mathbb{R}P^2;\mathbb{F}_2) \cong \mathbb{F}_2[\alpha]/(\alpha^3)$

polynomials in α, where $\alpha^2 = \gamma$, recall.

thm. $H^*(\mathbb{R}P^n;\mathbb{F}_2) = \mathbb{F}_2[\alpha]/(\alpha^{n+1})$

$H^*(\mathbb{R}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[\alpha]$

$K(\mathbb{Z}/2\mathbb{Z};1)$. group cohomology of $\mathbb{Z}/2\mathbb{Z}$.

eg. $H^*(T) = \Lambda_{\mathbb{Z}}[\alpha,\beta]$

$\alpha\smile\alpha = 0$ $\beta\smile\beta = 0$

$\alpha\smile\beta = -\beta\smile\alpha$

In general $H^*(T^n) = \Lambda[\alpha_1,\dots,\alpha_n]\dots$

Cup product is natural:

<u>Prop 3.10</u> $f : X \to Y$ map of top spaces
The induced maps

$$f^* : H^n(Y;R) \longrightarrow H^n(X;R)$$

satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.

Pf. Directly from definitions!

$$(f^\# \varphi \smile f^\# \psi)(\sigma)$$

$$= \underbrace{f^\# \varphi}_{\varphi f} \left(\sigma|_{[v_0, \dots, v_k]} \right) \cdot \underbrace{f^\# \psi}_{\psi f} \left(\sigma|_{[v_{k+1}, \dots, v_{k+l}]} \right)$$

$$= \varphi \left(f\sigma|_{[\dots]} \right) \cdot \psi \left(f\sigma|_{[v \dots]} \right)$$

$$= (\varphi \smile \psi)(f\sigma) = f^\#(\varphi \smile \psi)(\sigma).$$

Aside: Cross product aka external cup product

$$\begin{tikzcd}[column sep=small] & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & H^*(X \times Y) & \\ H^k(X) \arrow[ur, "p_1^*"] & & H^k(Y) \arrow[ul, "p_2^*"'] \end{tikzcd}$$

$$H^k(X;R) \times H^\ell(Y;R) \xrightarrow{\times} H^{k+\ell}(X \times Y; R)$$
$$(a,b) \longmapsto p_1^*(a) \smile p_2^*(b) =: a \times b$$

But note that since $\smile$ is distributive, the cross product is bilinear!

So by universal property of tensor products, there is a uniquely determined map (also called cross product)

$$H^k(X;R) \otimes H^\ell(Y;R) \xrightarrow{\times} H^{k+\ell}(X \times Y; R)$$
$$a \otimes b \longmapsto p_1^*(a) \smile p_2^*(b)$$
$$= a \times b$$

This is the map from the Künneth formula for cohomology!

Thm 3.16 $H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X \times Y; R)$ is an isom of rings if X and Y are CW cpxs and $H^k(Y;R)$ is a f.g. free R-module $\forall k$.

(no tor)

eg. $H^*(T^n;\mathbb{Z}) = H^*(\underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n \text{ copies}};\mathbb{Z})$.

$$H^*(S^1) = \Lambda[\alpha_1]$$

$$H^*(S^1 \times S^1) = \Lambda[\alpha_1] \otimes \Lambda[\alpha_2] \cong \Lambda[\alpha_1, \alpha_2]$$

Tensor product of graded-comm rings:

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|}\, aa' \otimes bb'.$$

Lecture 24

Now consider more regular topological spaces

<u>defn</u>

A <u>manifold of dimension n</u>, ie an <u>n-manifold</u>, is a Hausdorff space M where each point has an open nbhd homeo to $\mathbb{R}^n$

eg. line with two origins is not a manifold

- notice this definition doesn't allow for boundary; thats a manifold w/ ∂, where some points can have nbhds that look like $[0,\infty) \times \mathbb{R}^{n-1}$

- A compact manifold is called <u>closed</u> (as we have already discussed)

Alg-top interpretation of dimension of manifold:

let $x \in M$. Then the local homology group $H_i(M, M-\{x\}; \mathbb{Z})$ is nonzero only for $i=n$:

$$H_i(M, M-\{x\}; \mathbb{Z}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z})$$

by excising everything outside of a nbhd $\cong \mathbb{R}^n$ of x.

$$\cong \tilde{H}_{i-1}(\mathbb{R}^n - \{0\}; \mathbb{Z})$$

by LES of pair, and since $\mathbb{R}^n \simeq *$.

$$\cong \tilde{H}_{i-1}(S^{n-1}; \mathbb{Z})$$

$\mathbb{R}^n - \{0\} \simeq S^{n-1}$.

<u>Recall</u> LES:

$$\hookrightarrow H_i(\mathbb{R}^n - \{x\}) \longrightarrow H_i(\mathbb{R}^n) \longrightarrow H_i(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \to H_{i-1}(\mathbb{R}^n - \{x\}) \longrightarrow H_{i-1}(\mathbb{R}^n) \longrightarrow \cdots$$

Roadmap

There are many forms of duality for manifolds, all similar in flavor to the most primitive form below:

thm. (Poincaré duality)

- For a closed orientable n-mfld M^n, there are isomorphisms
  $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$. $\forall k$.
- If we drop orientability, we still have
  $H_k(M;\mathbb{Z}/2\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z}/2\mathbb{Z})$. $\forall k$.

Idea/Roadmap

- define orientation in terms of homology
- define a fundamental class for M in the top homology $[M]$
- define cap product
  $$\frown : C_k(X;R) \times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R)$$
- $[M] \frown -$ gives Poincaré Duality.

Remarks.

①

Fact The homology groups of a closed (recall, cpct) mfld are all finitely generated.

Thus using the UCT, we can also say the following in terms of only homology:

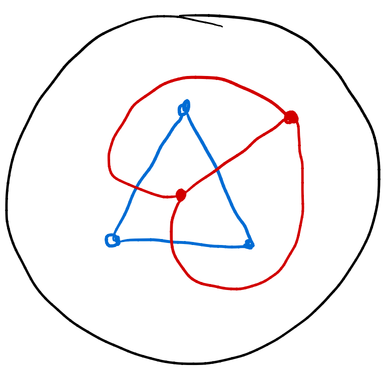
- M = orientable closed mfld.

Modulo torsion, $H_k(M;\mathbb{Z})$ and $H_{n-k}(M;\mathbb{Z})$ are isomorphic

②

geometric idea: Dual Cell structures

eg. S^2 planar web gives a cell structure
dual web gives a dual cell structure, representing cochains



blue vertices $\leftrightarrow$ red faces
blue edges $\leftrightarrow$ red edges
blue faces $\leftrightarrow$ red vertices

$$C_i \longleftrightarrow C^*_{n-i}$$

The interpretation that makes the most sense to me:
Morse / handle theory (only works for some manifolds)

Orientations & homology

Algebraic-topological defn of orientation:

<u>defn</u> An orientation of $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is a choice of generator of
$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \mathbb{Z}$$ as discussed previously

<u>rmk</u>. For any two points $x, y \in \mathbb{R}^n$, we can find a ball B containing both pts.
So the canonical isomorphisms
$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$$
allow us to propagate one choice of orientation to all points.
"local consistency"

<u>local orientations for manifolds</u>:

<u>defn</u> A local orientation of M at a point x is a choice of generator μ_x of the infinite cyclic group $H_n(M, M - \{x\})$.

<u>notation</u> Hatcher uses $H_n(X|A) := H_n(X, X-A)$ for shorter notation.
This is intuitive, since we're kind of looking at X restricted to A
We view $H_n(X|A)$ as the "local homology of X at A".

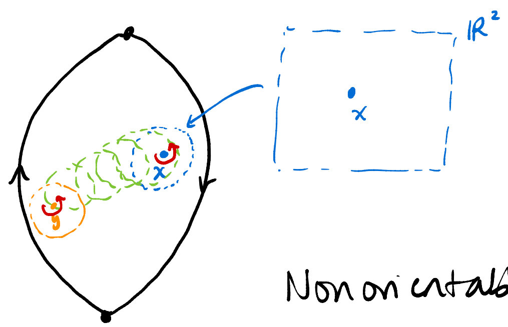
global orientations for manifolds:

defn An orientation of M^n is an function $x \mapsto \mu_x$ assigning to each point $x \in M$ a local orientation $\mu_x \in H_n(M|x)$ satisfying the "local consistency" condition:

each $x \in M$ has a nbhd homeo to $\mathbb{R}^n \subset M$ ← metric space (viewing $\mathbb{R}^n \subset M$) containing an open ball B of finite radius containing x s.t. all local orientations μ_y at points $y \in B$ are images of one generator μ_B of $H_n(M|B) \cong H_n(\mathbb{R}^n|B)$ under the natural maps $H_n(M|B) \longrightarrow H_n(M|y)$.

If an orientation exists for M, then M is orientable.

eg.



$H_2(\mathbb{R}^2|x) \cong H_1(S^1)$

Nonorientable because if you propagate μ_x the other way the two orientations at y disagree.

But we can extend the notion of orientability by using covers!
(need this to define fundamental class)