MAT 108 PS07

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Due Monday, May 18, 2026 at 9:00 pm on Gradescope

Instructions

There are three goals for this problem set:

- Maintain good mathematical writing form, as always.
- Reinforce familiarity with injectivity, surjectivity, bijectivity, and inverses of functions.
- Gain experience thinking about more abstract distance functions.

Here are some reminders about best practices:

- You absolutely must write in full, connected English sentences. Whenever possible, do not start a sentence with mathematical symbols. Do not use symbols like $\Rightarrow$, $\forall$, $\exists$, $\therefore$, etc. (except when explicitly discussing statements about logic).
- Your solution must be *neat*. You **will** be graded on style, which includes mathematical style and professionalism.
- It is your responsibility to mark where your solutions for each problem begins on Gradescope so that the TA and reader can properly grade your solutions.

Ungraded exercise

Read Section 10.2 to remind yourself of how to work with the absolute value function. We will be assuming these propositions without further comment.

Exercise 1

Determine whether the function is injective, surjective, bijective, or none of the above. Justify your assertions.

(a) $f : \mathbb{Z} \to \mathbb{Z}$, $n \mapsto n^2$

(b) $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$, $n \mapsto n^2$

(c) $f : \mathbb{N} \to \mathbb{N}$, $n \mapsto n^2$

(d) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 3x + 1$

(e) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $x \mapsto 3x + 1$

(f) $f : \mathbb{Z} \to \mathbb{Z}$, $x \mapsto 3x + 1$

Exercise 2

(a) Prove that $q : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, $[n] \mapsto [n]$, is surjective but not injective. Define two different right inverses j_1, j_2 for q.

(b) Prove that $i : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z}$, $[n] \mapsto [2n]$ is well-defined and injective, but not surjective. Define two different left inverses p_1, p_2 for i.

Exercise 3

Prove that if a function $f : A \to B$ of sets is bijective, then its inverse is unique.

Exercise 4

Prove that if $g \circ f$ is bijective, then g is surjective and f is injective.

Exercise 5

Recall that $d : X \times X \to \mathbb{R}$ is a *distance function* if all of the following hold:

- $d(x, y) = 0$ iff $x = y$
- $d(x, y) \geq 0$ for all $x, y \in X$ (we could have just written $d : X \times X \to \mathbb{R}_{\geq 0}$)
- $d(x, y) = d(y, x)$
- $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

Let $\mathbb{Z}[x]$ be the set of polynomials in the indeterminate variable x with coefficients in the integers. An arbitrary polynomial in $\mathbb{Z}[x]$ can be written as

$$a(x) = \sum_{i=0}^{\infty} a_i x^i$$

where there is some N such that for all $i > N$, $a_i = 0$.

Define a candidate metric function

$$\begin{aligned} \|\cdot\| : \mathbb{Z}[x] &\to \mathbb{R} \\ a(x) &\mapsto \sum_{i=0}^{\infty} |a_i| \cdot 10^{-i} \end{aligned}$$

and the associated candidate distance function

$$d(a(x), b(x)) = \|a(x) - b(x)\|.$$

(a) Prove that d is indeed a distance function.

(b) Find an infinite subset of polynomials $\{f_k\}_{k=1}^{\infty} \subset \mathbb{Z}[x]$ such that $\|f_k\| = 1$ for all k.