

MAT 108 PS08

Melissa Zhang

Due **Tuesday**, May 26, 2026 at 9:00 pm on Gradescope

Instructions

There are three goals for this problem set:

- Maintain good mathematical writing form, as always.
- Practice working with the rigorous definitions of ‘limit’, ‘convergence’, and related terms.
- Gain familiarity working with rational and irrational numbers, in preparation for the next unit on cardinality.

Here are some reminders about best practices:

- You absolutely must write in full, connected English sentences. Whenever possible, do not start a sentence with mathematical symbols. Do not use symbols like \Rightarrow , \forall , \exists , \therefore , etc. (except when explicitly discussing statements about logic).
- Your solution must be *neat*. You **will** be graded on style, which includes mathematical style and professionalism.
- It is your responsibility to mark where your solutions for each problem begins on Gradescope so that the TA and reader can properly grade your solutions.

Exercise 1

Prove that if a sequence of real numbers (x_k) converges, then the limit is unique.

Exercise 2

Let (a_k) be a sequence of real numbers that converges to $A \neq 0$. Prove that

$$\lim_{k \rightarrow \infty} \frac{1}{a_k} = \frac{1}{A}.$$

Exercise 3

In class, we proved the half of the Monotone Convergence Theorem for increasing sequences (Theorem 10.19). Prove the analogous statement for decreasing sequences, either by modifying the proof for Theorem 10.19 or by directly using Theorem 10.19.

Exercise 4

In class we proved that the sequence $(-1)^k$ diverges; it is bounded but not monotone. In this exercise, we consider sequences that are monotone but not bounded. For concreteness, we focus on increasing sequences.

We think of a positive real number as being “close to infinity” if it is “far from 0”. For example, the sequence (x_k) where $x_k = k$ approaches infinity because no matter how big a number M you give me, I can find an x_N such that all terms afterwards are greater than M .

- (a) Use the intuition above to write down a rigorous mathematical definition for $\lim_{k \rightarrow \infty} x_k = +\infty$. Your definition should start with the following phrase:

We say that a sequence (x_k) *approaches positive infinity* and write $\lim_{k \rightarrow \infty} x_k = +\infty$ if ...

- (b) Prove that $\lim_{k \rightarrow \infty} k^2 = +\infty$.
- (c) Prove that the sequence $x_k = (-1)^k k$ does *not* approach positive infinity.

Exercise 5

Suppose $r, s \in \mathbb{N}$ are both *not* perfect squares. Prove that $\sqrt{r} + \sqrt{s}$ is irrational.