Discussion Mock Quiz Problem 3

Melissa Zhang

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Exercise 1

(10 imaginary points) Prove that for all integers $n > 1$,

$$\sum_{j=1}^{n} \frac{1}{j^2} < 2 - \frac{1}{n}.$$

Proof. We induct on n.

(Base case $n = 2$.) Indeed

$$1 + \frac{1}{2^2} = 1 + \frac{1}{4} = 1.25 < 1.5 = 2 - \frac{1}{2}.$$

(Induction step.) Assume that for some $n > 1$, we have

$$\sum_{j=1}^{n} \frac{1}{j^2} < 2 - \frac{1}{n}.$$

By the induction hypothesis,

$$\sum_{j=1}^{n+1} \frac{1}{j^2} = \left(\sum_{j=1}^{n} \frac{1}{j^2}\right) + \frac{1}{(n+1)^2} \leq \left(2 - \frac{1}{n}\right) + \frac{1}{(n+1)^2} = 2 - \left(\frac{1}{n} - \frac{1}{(n+1)^2}\right).$$

But note that

$$\frac{1}{n} - \frac{1}{(n+1)^2} = \frac{(n+1)^2}{n(n+1)^2} - \frac{n}{n(n+1)^2} = \frac{n^2+n+1}{n(n+1)^2} > \frac{n^2+n}{n(n+1)^2} = \frac{1}{n+1}$$

Hence $\sum_{j=1}^{n+1} \frac{1}{j^2} \leq 2 - \left(\frac{1}{n} - \frac{1}{(n+1)^2}\right) \leq 2 - \frac{1}{n+1}$ as desired.

□

Rubric:

- *2 pts.* Style
- *2 pts.* Base case correct
- *1 pt.* Correct set-up for induction (base case + induction step and hypothesis)
- *5 pts.* Correct argument in induction step.