

MAT 108 Spring 2026
Exam 01 Solutions

1. (15 points + 5 style points)

Let $a, b \in \mathbb{N}$. Prove that

$$2a \leq b \leq a + 3 \text{ and } b \mid 3a^2 - 1$$

if and only if $a = 1$ and $b = 2$.

Hint: Start by using the inequalities to restrict the possible values of a .

For explicit numbers x, y , you don't need to prove to me whether $x \mid y$. For example, you may simply assert that 5 divides 25 and that 11 does not divide 25.

SOLUTION.

First, assume that $a = 1$ and $b = 2$. Then indeed

$$2a = 2 \leq 2 = b$$

and

$$b = 2 \leq 4 = 1 + 3 = a + 3.$$

Also, $3a^2 - 1 = 3 - 1 = 2$ is divisible by $b = 2$ since $2 = 1 \cdot 2$.

Conversely, assume that $2a \leq b \leq a + 3$ and $b \mid 3a^2 - 1$.

Since $2a \leq a + 3$, subtracting a from both sides, we have $a \leq 3$. Since $a \in \mathbb{N}$, this leaves us with three cases: $a \in \{1, 2, 3\}$.

If $a = 3$, then $6 \leq b \leq 6$ so $b = 6$. But $3a^2 - 1 = 27 - 1 = 26$ is not divisible by 6.

If $a = 2$, then $4 \leq b \leq 5$, so b is either 4 or 5. But $3a^2 - 1 = 12 - 1 = 11$, and neither 4 nor 5 divides 11.

Hence the only possibility is $a = 1$. Then $2 \leq b \leq 1 + 3 = 4$, so $b \in \{2, 3, 4\}$. Since $3a^2 - 1 = 2$, the only possibility is $b = 2$.

2. (15 points + 5 style points)

Prove that for all $k \in \mathbb{N}$, $2^{2k+1} + 1$ is divisible by 3.

SOLUTION.

We induct on k .

(Base case) For $k = 1$,

$$2^{2k+1} + 1 = 2^3 + 1 = 8 + 1 = 9$$

is divisible by 3 because $9 = 3 \cdot 3$.

(Induction step) Assume that for some $k \in \mathbb{N}$ we have

$$2^{2k+1} + 1 = 3j$$

for some $j \in \mathbb{Z}$. Now consider

$$2^{2(k+1)+1} + 1 = 2^{2k+1+2} + 1 = 4 \cdot 2^{2k+1} + 1 = 3 \cdot 2^{2k+1} + (2^{2k+1} + 1)$$

which, by the induction hypothesis is

$$= 3 \cdot 2^{2k+1} + 3j = 3(2^{2k+1} + j)$$

and is thus divisible by 3.

3. (15 points + 5 style points) Let $R, S, T \subset X$ for some set X . Show that

$$R \cap (S - T) = (R \cap S) - (R \cap T).$$

SOLUTION.

(There are many equivalent ways to write down this proof.)

We first show that $R \cap (S - T) \subseteq (R \cap S) - (R \cap T)$. If $x \in R \cap (S - T)$, then $x \in R$ and $x \in (S - T) = S \cap T^c$. In other words, $x \in R \cap S \cap T^c$. Hence $x \in R \cap S$ in particular. Also, since $x \notin T$, in particular $x \notin R \cap T$. Therefore $x \in (R \cap S) - (R \cap T)$.

We now show the reverse inclusion. Suppose that $x \in (R \cap S) - (R \cap T)$. This means that $x \in (R \cap S)$ and $x \notin R \cap T$. Since $x \in R \cap S$, in particular, $x \in R$. It remains to show that $x \in (S - T)$. Also since $x \in R \cap S$, in particular $x \in S$, so it suffices to show that $x \notin T$. Suppose by way of contradiction that $x \in T$. Since $x \in R$, it follows that $x \in R \cap T$. This contradicts our assumption that $x \notin R \cap T$.