MAT 108 Spring 2026

Exam 02 Solutions

1. Using the ε-N definition of convergence, prove that

$$\lim_{k\to\infty} \frac{k-5}{k} = 1.$$

SOLUTION.

Scratchwork Let $\varepsilon > 0$. We want to find N so that whenever $n \geq N$, we have

$$\left|\frac{k-5}{k} - 1\right| < \varepsilon.$$

The left-hand side is

$$\left|\frac{k-5}{k} - 1\right| = \left|\frac{-5}{k}\right| = \frac{5}{k}.$$

In order to have

$$\frac{5}{k} < \varepsilon,$$

we would need

$$k > \frac{5}{\varepsilon}.$$

Proof. Let $\varepsilon > 0$, and pick any $N > \frac{5}{\varepsilon}$, so that $\frac{5}{N} < \varepsilon$.

Then for all $n \geq N$, we have

$$\left|\frac{n-5}{n} - 1\right| = \left|\frac{-5}{n}\right| = \frac{5}{n} \leq \frac{5}{N} < \varepsilon.$$

□

2. Define a relation on the plane $\mathbb{R} \times \mathbb{R}$ by $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 + y_1 = x_2 + y_2$. Prove that this is an equivalence relation, and describe the equivalence classes.

SOLUTION.

To check that $\sim$ is an equivalence relation, we check that $\sim$ is reflexive, symmetric, and transitive.

- Since $x + y = x + y$, $\sim$ is reflexive.
- If $x_1 + y_1 = x_2 + y_2$, then $x_2 + y_2 = x_1 + y_1$, so $\sim$ is symmetric.
- If $x_1 + y_1 = x_2 + y_2$ and $x_2 + y_2 = x_3 + y_3$, then $x_1 + y_1 = x_3 + y_3$, so $\sim$ is transitive.

Given a point $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$, let $c = x_0 + y_0$. Its equivalence class is the set of all points (x, y) such that $x+y = c$. This is a line of slope -1, with y-intercept at $(0, c)$. The equivalence classes are all the lines in the plane of slope -1.

3. We say that two positive integers $p, q \in \mathbb{N}$ are *coprime* if for any $n \in \mathbb{Z}$,

$$\text{if } p \mid n \text{ and } q \mid n, \text{ then } pq \mid n.$$

Let p, q be coprime, and consider the assignment

$$f : \mathbb{Z}/pq\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$$
$$[n] \mapsto ([n], [n]).$$

Prove that f is a well-defined function, and that it is injective.

Solution.

We first check that f is well-defined. Suppose we have $n \equiv m \mod pq$. Then there exists $j \in \mathbb{Z}$ such that $n - m = jpq$. Therefore $n - m = (jq)p$ and $n - m = (jp)q$ so $n \equiv m \mod p$ and $n \equiv m \mod q$.

To check that f is injective, suppose we have $[n], [m] \in \mathbb{Z}/pq\mathbb{Z}$ such that

$$n \equiv m \mod p \qquad \text{and} \qquad n \equiv m \mod q.$$

Then there exist $j, k \in \mathbb{Z}$ such that

$$n - m = jp \qquad \text{and} \qquad n - m = kq.$$

Since p, q are coprime and both divide nm, we know $n-m$ is divisible by pq. Therefore $n \equiv m \mod pq$, so $[n] = [m]$ in $\mathbb{Z}/pq\mathbb{Z}$.