MAT 108: Final Exam Study Guide
Abridged Solutions

Melissa Zhang

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Have you tried the problems on your own first?

The main benefit of practice problems is that they help you figure out where you personally get stuck.

1. Show that there are no positive integer solutions $a, b \in \mathbb{N}$ to the equation $a^2 - b^2 = 10$.

   SOLUTION.

   There are many possible ways to prove this, and many ways of writing the same proof down. Here are two example proofs, just to show how different they can look.

   Example proof 1. Suppose by way of contradiction that there exist $a, b \in \mathbb{N}$ such that $a^2 - b^2 = 10$. Factoring the left-hand side, we have

   $$(a+b)(a-b) = 10.$$

   Since $a, b \in \mathbb{N}$, we have $a + b > 0$. Thus $a - b > 0$ as well, since otherwise their product would be negative. Furthermore, $a + b > a - b$ since $2b > 0$. The only positive factorizations of 10 (where the first factor is greater than or equal to the second) are $10 \cdot 1$ and $5 \cdot 2$.

   Case 1: Suppose $a + b = 10$ and $a - b = 1$. No such a, b exist because if $a - b = 10$, then $a \equiv b \mod 2$, which contradicts $a - b = 1$.

   Case 2: Suppose $a + b = 5$ and $a - b = 2$. Again, if $a - b = 2$, then $a \equiv b \mod 2$, contradicting $a + b = 5$. □

   Example proof 2. Consider the sequence

   $$x_n = (n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1.$$

   Now it is clear that x_n is strictly increasing. Observe that $x_5 = 6^2 - 5^2 = 36 - 25 = 11 > 10$. Hence if $a^2 - b^2 = 10$, then both a, b are at most 5. Furthermore, we must have $a > b$ since otherwise, if $b > a$, then $b^2 > ba > a^2$, a contradiction.

   Since $3^2 = 9 < 10$, we must also have $a \geq 4$. Thus $a \in \{4, 5\}$, which means $b^2 \in \{16 - 10, 25 - 10\} = \{6, 15\}$, which is impossible, because neither are perfect squares. Therefore there are no possible values $a, b \in \mathbb{N}$ for which $a^2 - b^2 = 10$. □

2. Show that there do not exist two integers $n, m \in \mathbb{Z}$ such that $n^4 - 4m = 2$.

   You may use the following lemma; make sure you make it clear where you use the lemma in your proof.

   Lemma A Let p be a prime. For $k \in \mathbb{Z}$, if $p \mid k^j$ for some $j \in \mathbb{N}$, then $p \mid k$.

   SOLUTION.

   Suppose, for contradiction, that there do exist $n, m \in \mathbb{Z}$ such that $n^4 - 4m = 2$. Taking this equation modulo 2, we see that n^4 is even. Since 2 is prime, we can use Lemma A to deduce that n is even. Therefore there exists some $a \in \mathbb{Z}$ such that $n = 2a$.

   Then

   $$n^4 - 4m = (2a) - 4m = 16a^4 - 4m = 4(4a^4 - m)$$

   is divisible by 4, but 2 is not divisible by 4. Contradiction.

3. Prove the following formula for all $n \in \mathbb{N}$:

$$\sum_{k=0}^{n}(2k+1) = (n+1)^2.$$

SOLUTION.

We induct on n. In the base case, $n = 1$, and indeed

$$(2 \cdot 0 + 1) + (2 \cdot 1 + 1) = 4 = (1+1)^2.$$

Now suppose the equation holds for some $n \in \mathbb{N}$; we will show it holds for $n+1$ as well. By the induction hypothesis,

$$\sum_{k=0}^{n+1}(2k+1) = \left(\sum_{k=0}^{n}(2k+1)\right) + 2(n+1) + 1 = (n+1)^2 + 2(n+1) + 1.$$

But this expression equals $(n+2)^2$, as desired:

$$(n+2)^2 = ((n+1)+1)^2 = (n+1)^2 + 2(n+1) + 1.$$

If you've forgotten about $(a+b)^2 = a^2 + 2ab + b^2$ or didn't recognize it, you could have also finished the proof by explicitly multiplying out these two polynomials.

4. Here are some more practice induction problems. (It is possible to prove these without induction, but we present them here for practice.)

   (a) Prove that for all $n \in \mathbb{N}$, $7 \mid (2^{n+2} + 3^{2n+1})$.

   (b) Prove that for all $n \in \mathbb{N}$, $5 \mid (11^n - 6)$.

   (c) Prove that for all $n \in \mathbb{N}$, $3 \mid n^3 - n$.

   (d) Show that for all $n \in \mathbb{N}$, $3^n \leq (n+3)!$.

SOLUTION.

Warning: The solutions below are not complete proofs! I've only included the induction step. The $*$ indicates where the induction hypothesis was used.

(a)

$$\begin{aligned} 2^{(n+1)+2} + 3^{2(n+1)+1} &= 2^{n+3} + 3^{2n+3} \\ &= 2 \cdot 2^{n+2} + 9 \cdot 3^{2n+1} \\ &= 2(2^{n+2} + 3^{2n+1}) + 7 \cdot 3^{2n+1} \\ &\overset{*}{=} 2(7j) + 7 \cdot 3^{2n+1} \\ &= 7(2j + 3^{2n+1}). \end{aligned}$$

(b)

$$\begin{aligned} 11^{n+1} - 6 &= 11 \cdot 11^n - 6 \\ &= 10 \cdot 11^n + (11^n - 6) \\ &\overset{*}{=} 10 \cdot 11^n + 5j \\ &= 5(2 \cdot 11^n + j). \end{aligned}$$

(c)

$$\begin{aligned}(n+1)^3-(n+1) &= n^3+3n^2+3n+1-n-1\\ &= (n^3-n)+3(n^2+n)\\ &\overset{*}{=} 3j+3(n^2+n)\\ &= 3(j+n^2+n).\end{aligned}$$

(d)

$$3^{n+1}=3\cdot 3^n \overset{*}{\leq} 3\cdot(n+3)! \leq (n+4)\cdot(n+3)! = (n+1+4)!.$$

5. Use the ε-N definition of the limit to show that

$$\lim_{n\to\infty}\frac{n!}{n^n}=0.$$

SOLUTION.

Scratchwork. Notice that $n! = n\cdot(n-1)\cdots 2\cdot 1 \leq n^{n-2}\cdot 2$. Therefore

$$\frac{n!}{n^n}\leq\frac{n^{n-2}\cdot 2}{n^n}=\frac{2}{n^2}.$$

So if we want

$$\left|\frac{n!}{n^n}-0\right|\leq\frac{2}{n^2}<\varepsilon,$$

we would need $n>\sqrt{\frac{2}{\varepsilon}}$.

Now notice how the scratchwork is incorporated into the official proof in a logical way.

Proof. Let $\varepsilon>0$. Choose $N\in\mathbb{N}$ such that $N>\sqrt{\frac{2}{\varepsilon}}$, so that for all $n\geq N$, we have $\frac{2}{n^2}<\varepsilon$. Since $n! = n\cdot(n-1)\cdots 2\cdot 1\leq n^{n-2}\cdot 2$, we have

$$\left|\frac{n!}{n^n}-0\right|=\frac{n!}{n^n}\leq\frac{n^{n-2}\cdot 2}{n^n}=\frac{2}{n^2}$$

For any $n\geq N$, we further have

$$\frac{2}{n^2}\leq\frac{2}{N^2}<\varepsilon.$$

□

6. Consider the following recursively defined sequence:

$$x_1=1,\qquad x_{n+1}=\frac{x_n}{2}+1\qquad\text{for } n\in\mathbb{N}.$$

(a) Find a closed expression (i.e. a non-recursive formula) for x_n.

(b) Show that (x_n) is bounded above and increasing. *Hint: Show boundedness first.*

(c) Prove that (x_n) converges and find its limit.

Solution.

(a) The first few terms are $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots$. We conjecture that $x_n = \frac{2^n-1}{2^{n-1}}$. There are other equivalent formulas, e.g.

$$x_n = 2 - \frac{1}{2^{n-1}}.$$

To prove this, we use induction.
Let $P(n)$ be the statement

$$x_n = \frac{2^n - 1}{2^{n-1}}.$$

The base case $P(1)$ is true, since

$$\frac{2^1 - 1}{2^0} = \frac{1}{1} = 1 = x_1.$$

Assuming $P(n)$ is true, we now show that $P(n+1)$ is true as well. By the recursion and inductive hypothesis,

$$x_{n+1} = \frac{x_n}{2} + 1 = \frac{2^n - 1}{2 \cdot 2^{n-1}} + 1 = \frac{2^n - 1}{2^n} + 1.$$

Rewriting 1 as $\frac{2^n}{2^n}$, we have that the above is

$$= \frac{2^n - 1 + 2^n}{2^n} = \frac{2^{n+1} - 1}{2^n} = \frac{2^{n+1} - 1}{2^{(n+1)-1}},$$

as desired.

(b) Abridged solution. Use induction to show that $x_n < 2$ for all n. Check the base case, then show that if $x_n < 2$, then $x_{n+1} < 2$ as well.
To show (x_n) is increasing, we need to show $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. Notice

$$x_{n+1} = \frac{x_n}{2} + 1 = \frac{x_n + 2}{2}$$

is the average of x_n and 2. Since $x_n < 2$, the average, x_{n+1}, is greater than x_n.
Alternatively, if you already had the blue formula, it is easier to show that (x_n) is increasing because as n grows, 2^{n-1} grows, so $\frac{1}{2^{n-1}}$ shrinks, so the whole expression $2 - \frac{1}{2^{n-1}}$ grows.

(c) Abridged solution. By the Monotone Convergence Theorem, since (x_n) is increasing and bounded above, it converges. From our intuition from part (b), we guess that the limit is 2. Rewrite the formula as

$$x_n = 2 - \frac{1}{2^{n-1}}$$

and show that $\frac{1}{2^{n-1}}$ limits to 0, by choosing N large enough so that

$$\frac{1}{2^{N-1}} < \varepsilon.$$

7. Prove that if $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

Solution. Let $x \in A \cup C$. Then $x \in A$ or $x \in C$. If $x \in A$, then $x \in B$ also, since $A \subset B$. So in either case, we have $x \in B$ or $x \in C$, i.e. $x \in B \cup C$.

8. Consider the following relation on $\mathbb{R}$:

$$x \sim y \qquad \text{iff} \qquad x - y \in \mathbb{Z}.$$

Prove that $\sim$ is an equivalence relation.

Solution. We check reflexivity, symmetry, and transitivity. Let $x, y, z \in \mathbb{R}$.

(Reflexivity.) Since $x - x = 0 \in \mathbb{Z}$, we have $x \sim x$.

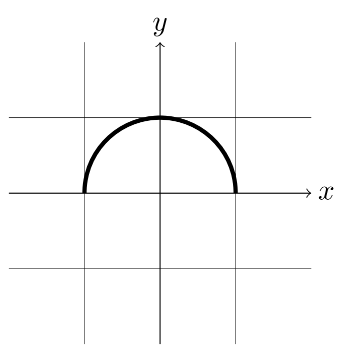
(Symmetry.) If $x \sim y$, then $x - y = z \in \mathbb{Z}$. Then $y - x = -z \in \mathbb{Z}$, so $y \sim x$.

(Transitivity.) If $x \sim y$ and $y \sim z$, then $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$. Therefore

$$x - z = x - z + y - y = (x - y) + (y - z) \in \mathbb{Z}$$

so $x \sim z$.

9. Define the function $f : [-1, 1] \to \mathbb{R}$ by $x \mapsto \sqrt{1 - x^2}$, whose graph is shown below:



Note that the slope of the graph is positive on $-1 < x < 0$ and negative on $0 < x < 1$.

Reminder: You must justify, i.e. prove, all of your answers to the following questions.

(a) Let $A = \mathrm{im} f$ denote the image of f. What is $\sup A$? What is $\inf A$?

(b) Is f an injective function? Is f a surjective function?

(c) Let (x_k) be an increasing sequence of numbers such that for all k, $0 \leq x_k \leq 1$. Prove that the sequence $(f(x_k))$ converges.

Solution.

(a) Since $x^2 \geq 0$, $f(x) \leq 1$. This is achieved at $x = 0$, so $\max(A) = 1$. Therefore $\sup(A) = 1$ as well.
Since $|x| \leq 1$ in the domain, $x^2 \leq 1$, so $f(x) \geq 0$. This is achieved at $x = \pm 1$, so $\min(A) = 0$. Therefore $\sup(A) = 0$ as well.

(b) The function f is not injective because $f(-1) = f(1) = 0$, but $-1 \neq 1$. It is also not surjective, because, for example, 2 is in the codomain $\mathbb{R}$, but $2 > \max(\text{im}(f))$, so there is no x in the domain for which $f(x) = 2$.

(c) On the interval $[0, 1]$, we see that $f(x)$ is decreasing. So if (x_k) is increasing, then $(f(x_k))$ is decreasing. Furthermore, $(f(x_k))$ is bounded below by 0. By the Monotone Convergence Theorem, $(f(x_k))$ converges.

10. Consider the sequence of real numbers $(x_k)_{k=1}^{\infty}$ given by

$$x_k = \frac{(-1)^k}{k}.$$

(a) Prove that $\lim_{k\to\infty} x_k = 0$.

(b) Let A be the set $\{x_k\}_{k\in\mathbb{N}}$. Find $\max(A), \min(A), \sup(A), \inf(A)$ (or prove that the quantity doesn't exist).

Solution.

(a) Let $\varepsilon > 0$, and pick some $N \in \mathbb{N}$ where $N > \frac{1}{\varepsilon}$, so that $\frac{1}{N} < \varepsilon$. Then for all $n \geq N$,

$$|x_n - 0| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

(b) Observe that the odd-indexed terms of (x_k) are negative, and the even-indexed terms are positive.

Since all negative numbers are less than any given positive number, to determine $\max(A)$ or $\sup(A)$, it suffices to consider the subset

$$A^+ = \{x_k \in A \ : \ k \text{ is even}\}$$

(which is nonempty because $x_2 \in A^+$).

Since $k + 2 > k$, $\frac{1}{k} > \frac{1}{k+2}$. So the sequence $x_2, x_4, x_6, \ldots$ is decreasing. Therefore the maximum is achieved at $x_2 = \frac{1}{n}$. So $\max(A) = \sup(A) = 2$.

Similarly, to find $\min(A)$ or $\inf(A)$, it suffices to consider the negative terms

$$A^- = \{x_k \in A \ : \ k \text{ is odd}\}.$$

Since $k + 2 > k$, $-\frac{1}{k} < -\frac{1}{k+2}$, so the sequence $x_1, x_3, x_5, \ldots$ is increasing. Therefore the minimum is achieved at $x_1 = -1$. So $\min(A) = \inf(A) = -1$.