MAT 108: Final Exam Study Guide

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1 Topics covered this quarter

- Properly writing mathematics
  - full sentences, proper punctuation
  - standard notation for written mathematics
  - axioms
  - if-then statements
  - uniqueness and existence
  - splitting into cases
- Induction
  - natural numbers
  - recursion
- Symbolic logic
  - for all, there exists, negation
  - implications
  - converse, inverse, contrapositive
  - proof by contradiction
- Set theory
  - sets, functions between sets
  - unions, intersections, complements
  - De Morgan's laws
  - Cartesian products
- Equivalence relations
  - criteria for equivalence relations
  - $\mathbb{Z}/n\mathbb{Z}$, modular arithmetic
- Functions
  - injections, surjections

  - bijections
  - domain, codomain, image, preimage
  - well-definedness

- Numbers, our main examples
  - $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R} - \mathbb{Q}$
  - Well-ordering principle
  - upper/lower bounds
  - Completeness Axiom

- Norms and distance functions
  - absolute value and distance between real numbers
  - other norms and how they define distance functions
  - criteria for distance functions / norms

- Limits
  - sequences
  - convergence, divergence
  - ε-N definition of a limit
  - Monotone Convergence Theorem
  - square roots

2 Practice Problems

These were discarded while writing the final exam.

1. Show that there are no positive integer solutions $a, b \in \mathbb{N}$ to the equation $a^2 - b^2 = 10$.

2. Show that there do not exist two integers $n, m \in \mathbb{Z}$ such that $n^4 - 4m = 2$.

   You may use the following lemma; make sure you make it clear where you use the lemma in your proof.

   Lemma A Let p be a prime. For $k \in \mathbb{Z}$, if $p \mid k^j$ for some $j \in \mathbb{N}$, then $p \mid k$.

3. Prove the following formula for all $n \in \mathbb{N}$:

$$\sum_{k=0}^{n} (2k+1) = (n+1)^2.$$

4. Here are some more practice induction problems. (It is possible to prove these without induction, but we present them here for practice.)

   (a) Prove that for all $n \in \mathbb{N}$, $7 \mid (2^{n+2} + 3^{2n+1})$.

   (b) Prove that for all $n \in \mathbb{N}$, $5 \mid (11^n - 6)$.

(c) Prove that for all $n \in \mathbb{N}$, $3 \mid n^3 - n$.

(d) Show that for all $n \in \mathbb{N}$, $3^n \leq (n+3)!$.

5. Use the ε-N definition of the limit to show that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

6. Consider the following recursively defined sequence:

$$x_1 = 1, \qquad x_{n+1} = \frac{x_n}{2} + 1 \qquad \text{for } n \in \mathbb{N}.$$

(a) Find a closed expression (i.e. a non-recursive formula) for x_n.

(b) Show that (x_n) is bounded above and increasing. *Hint: Show boundedness first.*

(c) Prove that (x_n) converges and find its limit.

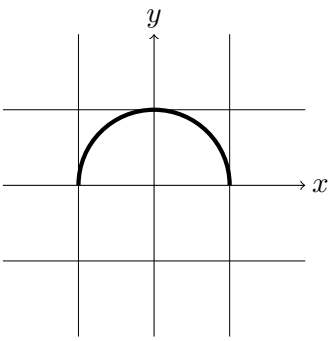
7. Prove that if $A \subseteq B$, then $A \cup C \subseteq B \cup C$.

8. Consider the following relation on $\mathbb{R}$:

$$x \sim y \qquad \text{iff} \qquad x - y \in \mathbb{Z}.$$

Prove that $\sim$ is an equivalence relation.

9. Define the function $f : [-1, 1] \to \mathbb{R}$ by $x \mapsto \sqrt{1 - x^2}$, whose graph is shown below:



Note that the slope of the graph is positive on $-1 < x < 0$ and negative on $0 < x < 1$.

Reminder: You must justify, i.e. prove, all of your answers to the following questions.

(a) Let $A = \text{im} f$ denote the image of f. What is $\sup A$? What is $\inf A$?

(b) Is f an injective function? Is f a surjective function?

(c) Let (x_k) be an increasing sequence of numbers such that for all k, $0 \leq x_k \leq 1$. Prove that the sequence $(f(x_k))$ converges.

10. Consider the sequence of real numbers $(x_k)_{k=1}^{\infty}$ given by

$$x_k = \frac{(-1)^k}{k}.$$

(a) Prove that $\lim_{k\to\infty} x_k = 0$.

(b) Let A be the set $\{x_k\}_{k\in\mathbb{N}}$. Find $\max(A), \min(A), \sup(A), \inf(A)$ (or prove that the quantity doesn't exist).