Selected solutions for PS01–PS04

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May 5, 2026

Exam 01 is on Friday, May 8, 2026, in our usual lecture room, at the usual lecture time. This exam covers material from PS01–PS04. Below are some selected solutions from these problem sets.

Disclaimer: These solutions are not meant to be representative of the material on the exam.

PS01 #3

a) Let $n \in \mathbb{Z}$ be any integer. To show that n divides 0, by definition, we need to show that there exists a $q \in \mathbb{Z}$ such that

$$0 = n \cdot q.$$

Setting $q = 0$, we get $n \cdot q = n \cdot 0 = 0$, so $n \mid 0$ indeed.

b) Let $m \neq 0$ be a nonzero integer. For all $q \in \mathbb{Z}$, $0 \cdot q = 0$, so $m \neq 0 \cdot q$. That is, for all $q \in \mathbb{Z}$, q cannot divide m.

c) Suppose $x \in \mathbb{Z}$ satisfies the property that for all $m \in \mathbb{Z}$, $mx = m$. In particular, $mx = m$ holds for all $m \neq 0$. In such cases (for example, when $m = 17$), we can use the cancellation axiom on the left-most and right-most sides of the equation

$$mx = m = m \cdot 1$$

to deduce that $x = 1$.

PS02 #3,4

First, we show by induction that for all $k \in \mathbb{N}$, $k \geq 1$ (i.e. $k = 1$ or $k > 1$).

Let $A = \{k \in \mathbb{Z} \ : \ k \geq 1\}$. (Base case) If $k = 1$, then $k \geq 1$, so $1 \in A$. (Induction step) Suppose $n \geq 1$. Since $(n + 1) - n = 1 \in \mathbb{N}$, we have $n + 1 > n > 1$. By transitivity, we have $n + 1 \geq 1$ as well, so $n + 1 \in A$.

By Axiom 2.15, since since $1 \in A$ and we have shown that if $n \in A$ then $n + 1 \in A$, we have $\mathbb{N} \subseteq A$, i.e. for any natural number k, $k \geq 1$.

We now show using proof by contradiction that there is no integer m such that $0 < m < 1$. Note that this means $m \neq 0$ and $m \neq 1$.

By way of contradiction, suppose that there is an integer m such that $0 < m < 1$. Then

$$m - 0 = m \in \mathbb{N} \qquad \text{and} \qquad 1 - m \in \mathbb{N}.$$

In the previous paragraph, we showed that, since $m \in \mathbb{N}$, $m \geq 1$. Since $m \neq 0$, we must have $m > 1$, i.e. $m - 1 \in \mathbb{N}$. But we also have $1 - m \in \mathbb{N}$. But this is impossible by Proposition 2.2: since $m \neq 1$, both $m - 1$ and $1 - m = -(m - 1)$ are not 0, so only one of them can be in $\mathbb{N}$.

PS03 #2

We claim that for all $n \in \mathbb{N}$, $S_n = \frac{2^n-1}{2^n}$. We induct on n.

(Base case) For $n = 1$, indeed $S_n = \frac{1}{2} = \frac{2^1-1}{2^1}$.

(Induction step) Assume that the induction hypothesis holds, i.e. for some n we have $S_n = \frac{2^n-1}{2^n}$. We will show that $S_{n+1} = \frac{2^{n+1}-1}{2^{n+1}}$ First observe that

$$S_{n+1} = \sum_{k=1}^{n+1} \frac{1}{2^k} = \left(\sum_{k=1}^{n} \frac{1}{2^k} \right) + \frac{1}{2^{n+1}} = S_n + \frac{1}{2^{n+1}}.$$

By the induction hypothesis, the above is

$$= \frac{2^n - 1}{2^n} + \frac{1}{2^{n+1}} = \frac{2(2^n-1)}{2(2^n)} + \frac{1}{2^{n+1}} = \frac{2^{n+1}-2}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{2^{n+1}-1}{2^{n+1}}$$

as desired.

PS03 #5

(Recall that these are potentially nonsense statements.)

a) There exists a cubic polynomial without any real roots.

b) Either G is not normal or H is not regular.

c) For all 0, there exists x such that $x + 0 \neq x$.

d) The newspaper article was accurate or entertaining.

e) If $\gcd(m, n)$ is odd, then both m and n are even.

f) There exists H, N, G such that either H/N is a normal subgroup of G/N but H is not a normal subgroup of G, or H is a normal subgroup of G but H/N is not a normal subgroup of G/N.

g) There exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $n \geq N$ such that $|a_n - L| \leq \varepsilon$.

PS03 #6

a) (Inverse) If I don't have three arms, then the sky isn't blue.
(Converse) If the sky is blue, then I have three arms.
(Contrapositive) If the sky isn't blue, then I don't have three arms.

b) (Inverse) If we can't find two cats of the same color, then there exist cats of different colors.
(Converse) If all cats are the same color, then we can find two cats that are the same color.
(Contrapositive) If not all cats are the same color, then we cannot find two cats that are the same color.

c) First note that the statement can be rewritten as "If it rains, then I wear boots."
(Inverse) If it doesn't rain, then I don't wear boots.
(Converse) If I wear boots, then it rains.
(Contrapositive) If I don't wear boots, then it doesn't rain.

PS04 #2

Let n be a natural number. Let $P(n)$ be the statement the following statement:

For all $m, k \in \mathbb{N}$ such that $m \geq 2$ and $k \geq 1$ (so that $m-1 \in \mathbb{N}$) and $n = m+k$,

$$f_{m+k} = f_{m-1}f_k + f_m + f_{k+1}.$$

We will induct on n.

In the base case, $m = 2$ and $k = 1$, so we consider $n = 3$. Indeed,

$$f_{m-1}f_k + f_m + f_{k+1} = f_1 \cdot f_1 + f_2 \cdot f_2 = 1 \cdot 1 + 1 \cdot 1 = 2 = f_3.$$

Now consider a fixed $n > 3$, and assume that $P(j)$ holds for all $3 \leq j \leq n$. We want to show that $P(n+1)$ holds. Observe that it suffices to check that for all m, k such that $m + k = n$, the following two equations hold:

$$f_{(m+1)+k} = f_m f_k + f_{m+1} f_{k+1} \tag{1}$$

and

$$f_{m+(k+1)} = f_{m-1} f_{k+1} + f_m f_{k+2}. \tag{2}$$

To prove Equation 1, we begin from the right-hand side of the equation and use the recurrence relation:

$$\begin{aligned} f_m f_k + f_{m+1} f_{k+1} &= (f_{m-1} + f_{m-2}) f_k + (f_m + f_{m-1}) f_{k+1} \\ &= (f_{m-1} f_k + f_{m-2} f_k) + (f_m f_{k+1} + f_{m-1} f_{k+1}). \end{aligned}$$

After reordering the terms, this is

$$= (f_{m-1} f_k + f_m f_{k+1}) + (f_{m-2} f_k + f_{m-1} f_{k+1}).$$

By the induction hypothesis, $P(n)$ is true, so

$$(f_{m-1} f_k + f_m f_{k+1}) = f_{m+k} = f_n,$$

and $P(n-1)$ is true, so

$$(f_{m-2} f_k + f_{m-1} f_{k+1}) = f_{(m-1)+k} = f_{n-1}.$$

Therefore

$$f_m f_k + f_{m+1} f_{k+1} = f_n + f_{n-1} = f_{n+1} = f_{m+k+1}.$$

This is Equation 1.

To prove Equation 2, we again begin on the right-hand side. We similarly expand the two terms using the recurrence relation:

$$\begin{aligned} &f_{m-1} f_{k+1} + f_m f_{k+2} \\ &= f_{m-1}(f_k + f_{k-1}) + f_m (f_{k+1} + f_k) \\ &= (f_{m-1} f_k + f_{m-1} f_{k-1}) + (f_m f_{k+1} + f_m f_k) \end{aligned}$$

and then combine terms using the induction hypothesis:

$$\begin{aligned} &= f_{m-1} f_k + f_m f_{k-1} + f_{m-1} f_{k-1} + f_m f_{k+1} \\ &= (f_m f_{k-1} + f_{m-1} f_{k-1}) + (f_{m-1} f_k + f_m f_{k+1}) \\ &= f_{m+k} + f_{m+(k-1)}. \end{aligned}$$

This proves Equation 2.

PS04 #4

(a) The statement $A - (B \cup C) = (A - B) \cup (A - C)$ is false. As a counterexample, consider the sets

$$A = \{1, 2, 3\}$$
$$B = \{2\}$$
$$C = \{3\}.$$

Then $A - (B \cup C) = \{1\}$ whereas $(A - B) \cup (A - C) = \{1, 3\} \cup \{1, 2\} = \{1, 2, 3\}$.

(b) The statement $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ is also false. The intuition is that an element in $C \times B$ is guaranteed to be in the RHS, but not guaranteed to be in the LHS. So, we construct a counterexample where this is the case.

Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, and $D = \{4\}$. Then

$$(A \times B) \cup (C \times D) = \{(1, 2)\} \cup \{(3, 4)\} = \{(1, 2), (3, 4)\}.$$

On the other hand,

$$(A \cup C) \times (B \cup D) = \{1, 3\} \times \{2, 4\} = \{(1, 2), (1, 4), (3, 2), (3, 4)\}.$$

(c) The statement $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ is true, and we prove this using double inclusion.

First, we'll show that the LHS is contained in the RHS. Let $(x, y) \in (A \times B) \cap (C \times D)$. Since $(x, y) \in A \times B$, we know that $x \in A$ and $y \in B$. Similarly, $(x, y) \in (C \times D)$, so $x \in C$ and $y \in D$. Therefore $x \in A \cap C$ and $y \in B \cap D$, so $(x, y) \in (A \cap C) \times (B \cap D)$.

Second, we'll show that the RHS is contained in the LHS. Let $(u, v) \in (A \cap C) \times (B \cap D)$. Then u is in both A and C, and v is in both B and D. Therefore (u, v) is in both $A \times B$ and $C \times D$, so indeed $(u, v) \in (A \times B) \cap (C \times D)$.

MAT 108 Spring 2026
Exam 01

By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else. I understand that any suspicions of collaboration, cheating, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.

Name (sign): ___________________________ Name (print): ___________________________

Problem	Points Possible	Points Earned
Q1	15	
Q1 style	5	
Q2	15	
Q2 style	5	
Q3	15	
Q3 style	5	
Total:	60	

- You have **45 minutes** to complete this exam. If you are done early, you may leave after handing me your exam packet.
- Tear off the last page of this packet to use as scratch paper. It is **highly recommended** that you use scratch paper to figure out how you want to structure your proof, and then write down your final answer in the test packet.
- **You will be graded on both mathematical reasoning and on writing style.** In particular, your proofs must be in full sentences and **neat**, emulating the style of a typeset mathematical document. You will only be graded on the written proofs in your exam packet.
- Per the syllabus, **no electronics** may be used at any time during the exam period. All electronics must be stored inside a bag or at the front of the classroom. **No notes, texts, or other materials are allowed.**