Selected solutions for PS05–PS08

Melissa Zhang

May 30, 2026

Exam 02 is on Monday, June 1, 2026, in our usual lecture room, at the usual lecture time. Bryce Thompson will be proctoring the exam. This exam covers material from PS05–PS08. Below are some selected solutions from these problem sets. *Disclaimer: These solutions are not meant to be representative of the material on the exam.*

At the end of this document I've also attached

- the cover Exam 02, which includes all the exam policies (same as for Exam 01), and
- a redacted page from the exam, which shows how you'll be asked or organize our work for the limits problem. The purpose is for the grader to see how you arrived at your choice of N.

PS05 #2

(a) This is not an equivalence relation because it is not reflexive: $0 \not\sim 0$ since $0 \cdot 0 = 0$ is not positive. This *would* be an equivalence relation on $\mathbb{Z} - \{0\}$.

(b) No, this relation fails transitivity. For example, $1 \sim 2$ since $1 \mid 2$, and $1 \sim 3$ since $1 \mid 3$. But $2 \not\sim 3$ since neither $2 \mid 3$ nor $3 \mid 2$ is true.

(c) Yes, this is an equivalence relation. To see this, let $x, y, z \in \mathbb{Z}$.

- (Reflexivity.) $x = x$, so $x \sim x$.
- (Symmetry.) If $x \sim y$, then $x = y$ or $x = -y$. In the first case, we then have $y = x$. In the second case, then $y = -x$. In either case, we have $y \sim x$.
- (Transitivity.) If $x \sim y$ and $y \sim z$, then there m, n such that $x = (-1)^m y$ and $y = (-1)^n z$. Hence $x = (-1)^m(-1)^n z = (-1)^{m+n} z$. Hence $x = \pm z$, so we have $x \sim z$.

Alternatively, you could first show that $x \sim y$ if and only if $|x| = |y|$. This makes checking symmetry and transitivity easier.

The equivalence classes are

- $[0] = \{0\}$ and
- $[n] = \{-n, n\}$ for $n \neq 0$.

PS05 #4

Suppose we have $a, a' \in [a]$ and $b, b' \in [b]$. Then there exist $j, k \in \mathbb{Z}$ such that

$$a' = a + jn \qquad and \qquad b' = b + kn.$$

Then

$$a'b' = (a + jn)(b + kn) = ab + jnb + akn + jkn^2 = ab + n(jb + ak + jkn)$$

so $a'b' \equiv ab \mod n$. Therefore $[a'b'] = [ab]$.

PS06 #2

(a) By the Completeness Axiom, because B is bounded above, it has a supremum $\sup(B) \in \mathbb{R}$. Since $\sup(B)$ is the least upper bound, it is in particular an upper bound. So for all $b \in B$, $b \le \sup(B)$. Since $A \subseteq B$, any $a \in A$ is also in B, so in particular $a \le \sup(B)$. Therefore $\sup(B)$ is an upper bound for A. Since $\sup(A)$ is the *least* upper bound for A, we conclude that $\sup(A) \le \sup(B)$.

(b) We first prove Proposition 8.53:

Proposition. Every nonempty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.

Proof. Let $\emptyset \neq S \subseteq \mathbb{R}$ and suppose S is bounded below by $b \in \mathbb{R}$. This means that for all $x \in S$, we have $x \ge b$.

Now consider the related set

$$S' := \{-x \ : \ x \in S\}.$$

Note that S' is nonempty, since there is at least one $x \in S$, and so $-x \in S'$. Moreover, for any $-x \in S'$, we have $-x < -b$ because $x > b$. Hence S' is bounded above. By the Completeness Axiom, S' has a least upper bound $\sup(S')$.

We claim that $-\sup(S')$ is the greatest lower bound for S. To see this, let b be any lower bound for S. Then $-b$ is an upper bound for S', and so $-b \ge \sup(S')$ so $b \le -\sup(S')$.

Therefore $\inf(S) = \sup(S') \in \mathbb{R}$. □

Now we use the same argument as in part (a). Given any $a \in A$, $a \in B$, and so $a \ge \inf(B)$. Therefore $\inf(B)$ is a lower bound for A, and so the greatest lower bound is no less than $\inf(B)$: $\inf(A) \ge \inf(B)$.

PS07 #1

(Abridged solutions.)

(a) Not injective: $f(-1) = f(1) = 1$. Not surjective: 2 is not a perfect square.

(b) Still neither; counterexamples in (a) still hold.

(c) Injective: If $n^2 = m^2$, then $n = \sqrt{n^2} = \sqrt{m^2} = m$. Not surjective: 2 is not a perfect square.

(d) Bijective: f has inverse function $g(x) = \frac{x-1}{3}$.

(e) Injective: If $3x + 1 = 3x' + 1$, then $3x = 3x'$, so $x = x'$. In other words, f has left inverse g as defined in part (d), but with restricted codomain. Not surjective: There is no positive $x \in \mathbb{R}_{\ge 0}$ such that $3x + 1 = 0$; x would have to be $-\frac{1}{3}$.

(f) Injective: Same proof as in part (e). Not surjective: Same counterexample as in part (e).

PS07 #4

Suppose we have sets A, B, C and functions $f : A \to B$ and $g : B \to C$ such that $g \circ f : A \to C$ is bijective.

Given any $c \in C$, there is some $a \in A$ such that $g \circ f(a) = c$. Let $b = f(a)$. Then $g(b) = c$. Since c was arbitrary, we have shown that g is surjective.

Given $a, a' \in A$, suppose $f(a) = f(a')$. Then $g(f(a) = g(f(a'))$. Since $g \circ f$ is injective, we have $a = a'$. So f is also injective.

PS08 #2

Scratchwork: We know that for any $\varepsilon > 0$, we can find some N such that for all $n \geq N$, we have

$$|a_n - A| < \varepsilon.$$

Let $\eta > 0$. We want n big enough so that

$$\left| \frac{1}{a_n} - \frac{1}{A} \right| < \eta.$$

The left-hand side can be rewritten as

$$\frac{|A - a_n|}{|a_n A|}.$$

For this fraction to be small, we want the numerator to be small and the denominator to be big.

- The numerator is easily related to the limit we are given.
- By Proposition 10.23 (ii), we know that $\lim_{k\to\infty} a_k A = A^2$, so for any $\varepsilon' > 0$, there exists some N' such that for all $n \geq N'$, $|a_n A - A^2| < \varepsilon'$.

So we can just make sure the denominator stays bigger than, say, $\frac{A^2}{2} > 0$, and then make the numerator small enough so that the ratio ends up less than η. (Note that $|\frac{A^2}{2} - A^2| = \frac{A^2}{2}$.)

Which ε will we need to use? If our n is already big enough so that $|a_n A| > \frac{A^2}{2}$, i.e.

$$\frac{|A - a_n|}{|a_n A|} < \frac{|A - a_n|}{|\frac{1}{2} A^2|} \qquad \text{(which we want } < \eta),$$

then we just need to make sure n is big enough so that

$$|A - a_n| < \frac{\eta A^2}{2}.$$

To summarize,

- we use $\lim_{k\to\infty} a_k = A$ with $\varepsilon = \frac{\eta A^2}{2}$ to get N and
- we use $\lim_{k\to\infty} |a_k A| = A^2$ with $\varepsilon' = \frac{A^2}{2}$ to get N',

and we can let $M > \max N, N'$. Then for all $n \geq M$, we should be guaranteed that the ratio is sufficiently small ($< \eta$). We now write the proof, which also helps us check our work.

Proof:

Proof. Let $\eta > 0$. Since $\lim_{k\to\infty} a_k = A$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n - A| < \frac{\eta A^2}{2}.$$

Since $\lim_{k\to\infty} |a_k A| = A^2$, there exists $N' \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|a_n A - A^2| < \frac{A^2}{2},$$

or equivalently, $|a_n A| > \frac{A^2}{2}$.

Now pick some $M > \max\{N, N'\}$. Then for all $n \geq M$,

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| = \frac{|A - a_n|}{|a_n A|} < \frac{|A - a_n|}{\frac{1}{2}A^2} < \frac{\frac{1}{2}\eta A^2}{\frac{1}{2}A^2} = \eta.$$

□

PS08 #3

Here is one possible proof.

Let $(x_k)_{k=1}^{\infty}$ be a bounded, decreasing sequence; this means that $x_k \geq x_{k+1}$. Now consider the sequence $(-x_k)$; this is increasing because $-x_k \leq -x_{k+1}$. If (x_k) is bounded between m and M (i.e. $m \leq x_k \leq M$ for all k), then $(-x_k)$ is bounded between $-M$ and m.

By Theorem 10.19, $(-x_k)$ converges to some limit L. In other words, for any $\varepsilon > 0$, there exists N such that for all $n \geq N$, we have $|-x_n - L|$. But

$$|-x_n - L| = |x_n - (-L)|,$$

so $\lim_{k\to\infty} x_n = -L$.

MAT 108 Spring 2026

Exam 02

By providing my signature below I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else. I understand that any suspicions of collaboration, cheating, or otherwise violating the Student Code of Conduct will be forwarded to the Student Judicial Board.

Name (sign): ______________________ Name (print): ______________________

Problem	Points Possible	Points Earned
Q1	15	
Q1 style	5	
Q2	15	
Q2 style	5	
Q3	15	
Q3 style	5	
Total:	60	

- You have **45 minutes** to complete this exam. If you are done early, you may leave after handing me your exam packet.
- Tear off the last page of this packet to use as scratch paper. It is **highly recommended** that you use scratch paper to figure out how you want to structure your proof, and then write down your final answer in the test packet.
- **You will be graded on both mathematical reasoning and on writing style.** In particular, your proofs must be in full sentences and **neat**, emulating the style of a typeset mathematical document. You will only be graded on the written proofs in your exam packet.
- Per the syllabus, **no electronics** may be used at any time during the exam period. All electronics must be stored inside a bag or at the front of the classroom. **No notes, texts, or other materials are allowed.**

2. (15 points + 5 style points)

   Using the ε-N definition of convergence, prove that

   REDACTED

   Scratchwork (not graded):

   Proof (graded):